On the relationship between

Electromagnetic curvature and acceleration of charges

by Yaron Hadad

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The Problem of Self-Force

Problem #1: Dynamics of a particle in a known external field

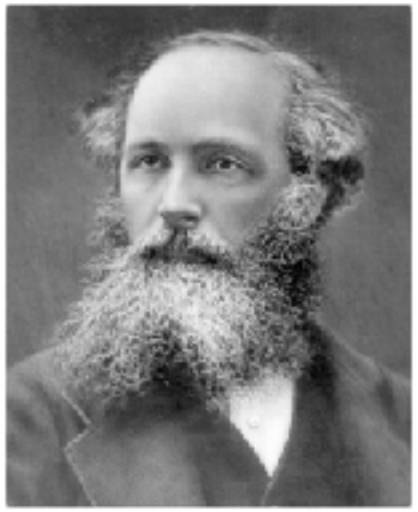
$m\dot{u}^\alpha = -eF_{\rm ext}^{\alpha\beta}u_\beta$ well-posed for *any* external field $F_{\rm ext}^{\alpha\beta}$

Problem #2: Dynamics of the field for known currents

$\partial_\alpha F^{\alpha\beta} = 4\pi J^\beta$ well-posed if $\partial_\alpha J^\alpha = 0$

Also well-defined for a point particle

$$J^\alpha(x) = -eu^\alpha \delta(x - z(t))\frac{d\tau}{dt}$$



Maxwell 1865

Problem #3 = #1 + #2: The coupled system

Mathematically ill-defined.

The Problem of Radiation-Reaction

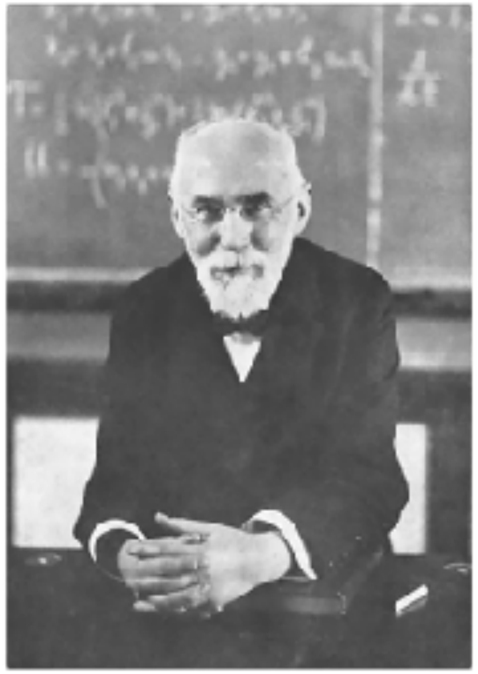
The Lorentz Force (LF) Eq: $m\dot{u}^\alpha = -eF^{\alpha\beta}_{\text{ext}}u_\beta$

The rate at which energy is radiated away from the electron is

$$\mathcal{R} = -m\tau_0\dot{u}^\alpha\dot{u}_\alpha$$

$$\tau_0 = \frac{2}{3}\frac{e^2}{m} = 6.24 \times 10^{-24}\,\text{s}$$

$\Longrightarrow$ an accelerating charge loses energy.



Lorentz 1892

This effect is not included in the Lorentz Force equation. The rate at which energy-momentum is emitted by radiation:

$$\frac{dP^\alpha}{d\tau} = \mathcal{R}u^\alpha$$

A Plentitude of Models…

Lorentz-Abraham-Dirac (1938)	$m\dot{u}^\alpha = \underbrace{-eF^{\alpha\beta}u_\beta}_{\text{Lorentz force}} + \underbrace{m\tau_0 \left[\overbrace{\ddot{u}^\alpha}^{\text{Schott}} + \overbrace{\dot{u}^2 u^\alpha}^{\text{radiation-reaction}}\right]}_{\text{self force}}$
Landau-Lifshitz (1952)	$m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta - e\tau_0\left[F^{\alpha\beta}{}_{,\gamma}u_\beta u^\gamma - \frac{e}{m}\left(F^{\alpha\beta}F_{\beta\gamma}u^\gamma - F^{\beta\gamma}F_{\gamma\delta}u^\delta u_\beta u^\alpha\right)\right]$
	$\dots u_\gamma u^\alpha\Big]$
	$\dots u_\gamma u^\alpha\big]$
	$\dots)u_\beta(\tau)$
Caldirola-Yaghjian (1992)	$m\dot{u}^\alpha = -eF^{\alpha\beta}(\tau)u_\beta(\tau) - \frac{m}{\tau_0}\left[u^\alpha(\tau-\tau_0) - u^\alpha(\tau)u_\beta(\tau)u^\beta(\tau-\tau_0)\right]$

Effects of Radiation-Reaction in Relativistic Laser Acceleration

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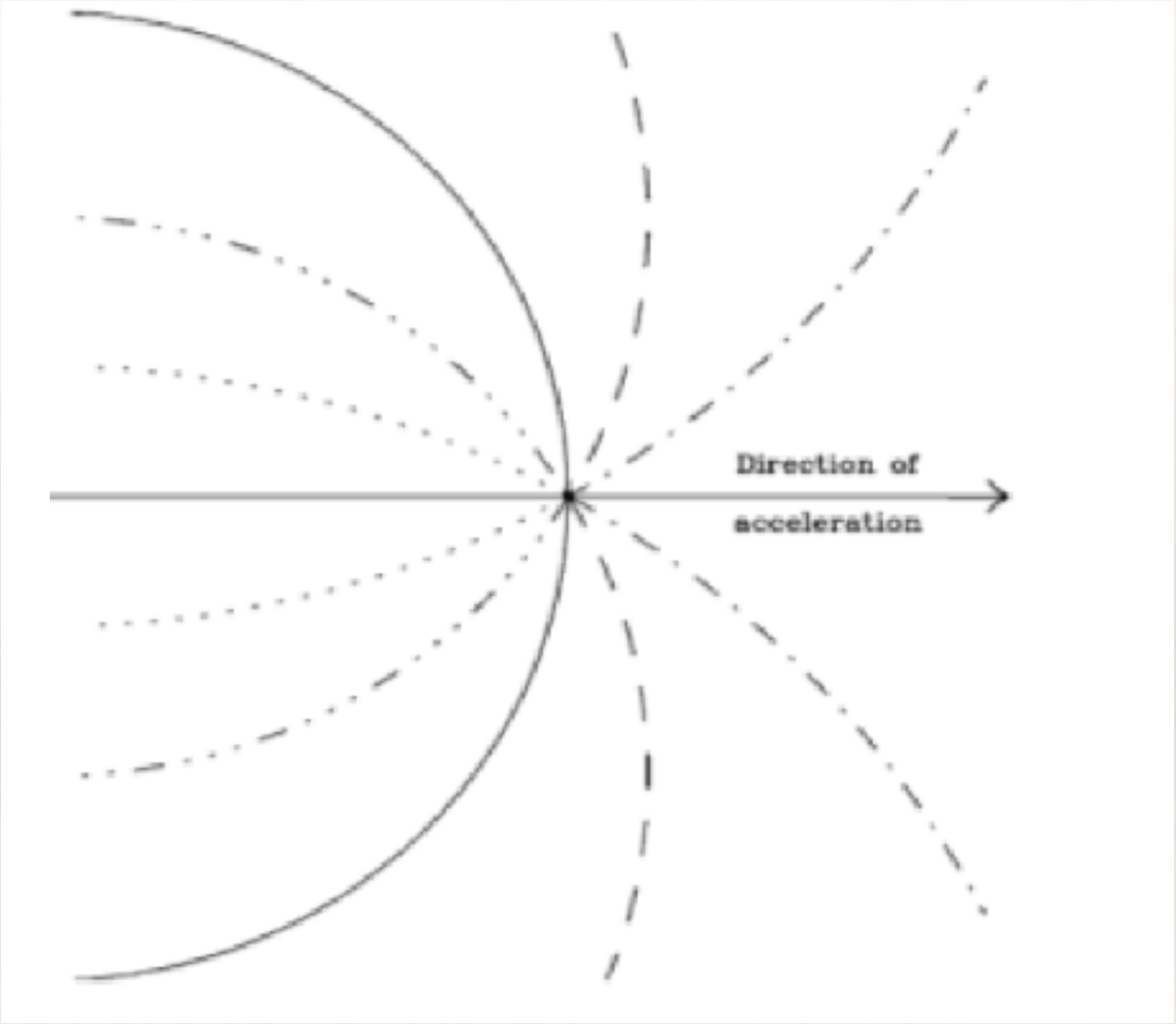
(Dated: 14 November, 2010)

More equations: Prigogine-Henin (1962), Nodvik (1964), Teitelboim (1970), Gonzales-Gascon (1976), Petzold-Sorg (1977), Ford-O'Connell (1991), Sokolov et al. (2009), Hammond (2011), Cabo-Castineiras (2013), more?

Field Lines

Michael Faraday



Charge acceleration $\overset{?}{\Longleftrightarrow}$ Field lines curvature

Analytical Field Lines Curvature

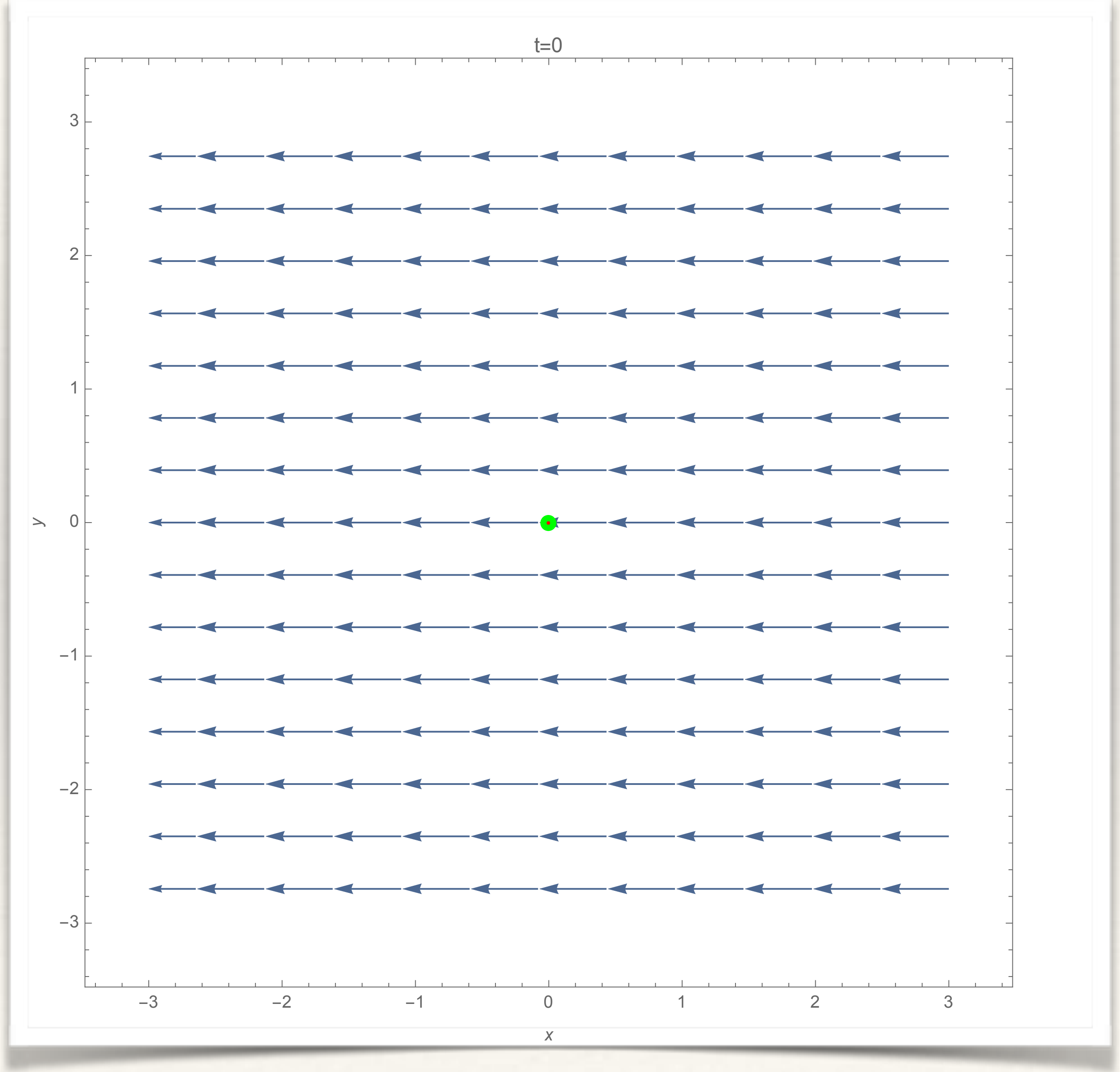
The electrostatic field line eq:

$$\frac{d\vec{\gamma}}{ds} = \vec{E}(\vec{\gamma}(s)) \qquad \vec{\gamma}(0) = \vec{\gamma}_0$$

The curvature of a curve $\vec{\gamma}$

$$\kappa = \frac{|\vec{\gamma}' \times \vec{\gamma}''|}{|\vec{\gamma}'|} = \frac{|\vec{E} \times (\vec{E} \cdot \nabla)\vec{E}|}{|\vec{E}|^3}$$

$\vec{E}_{\text{ext}}$ - External Field Lines (E=constant)



No field lines curvature

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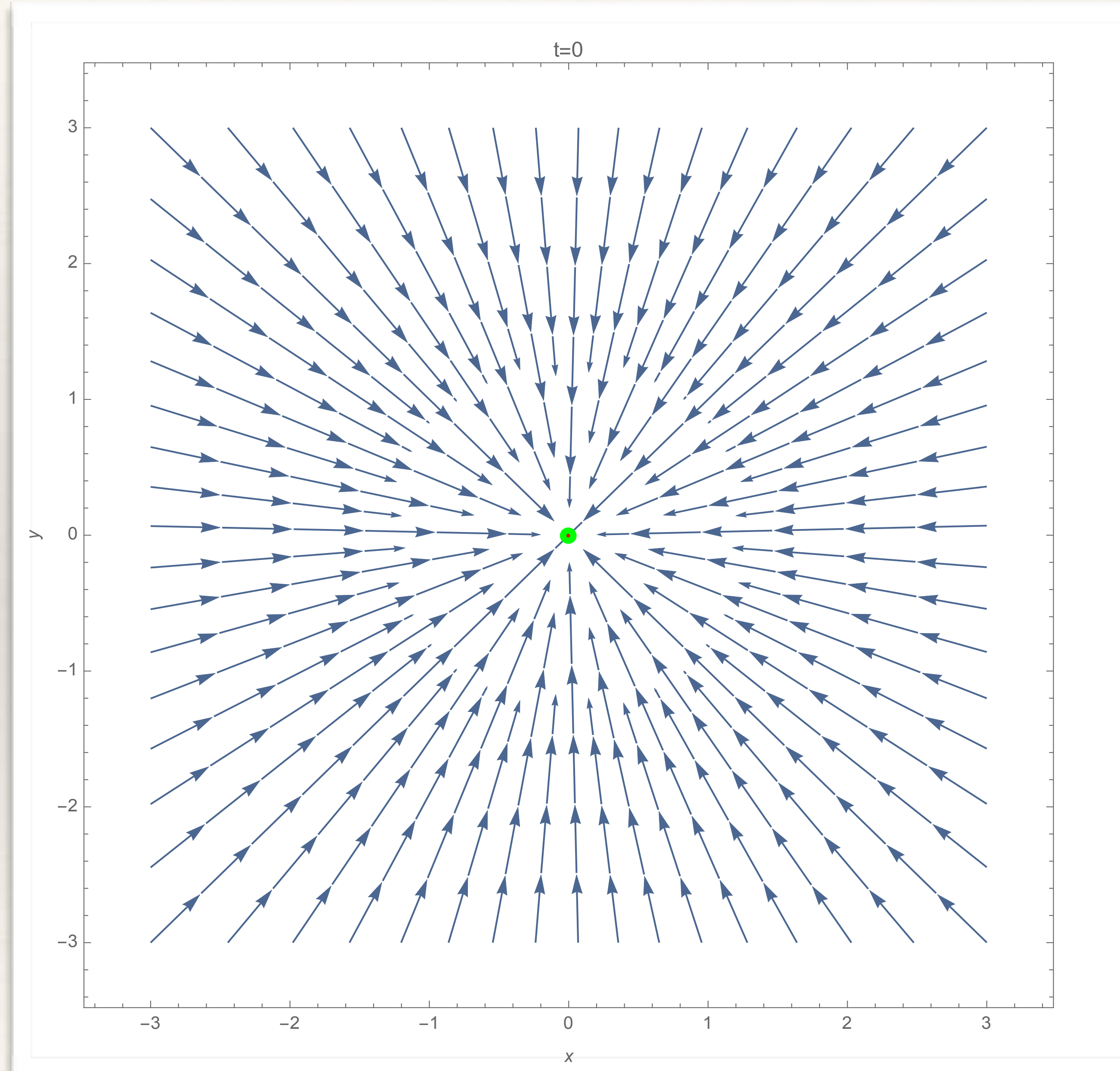
$\vec{E}_{\mathrm{self}}$ - Self Field Lines



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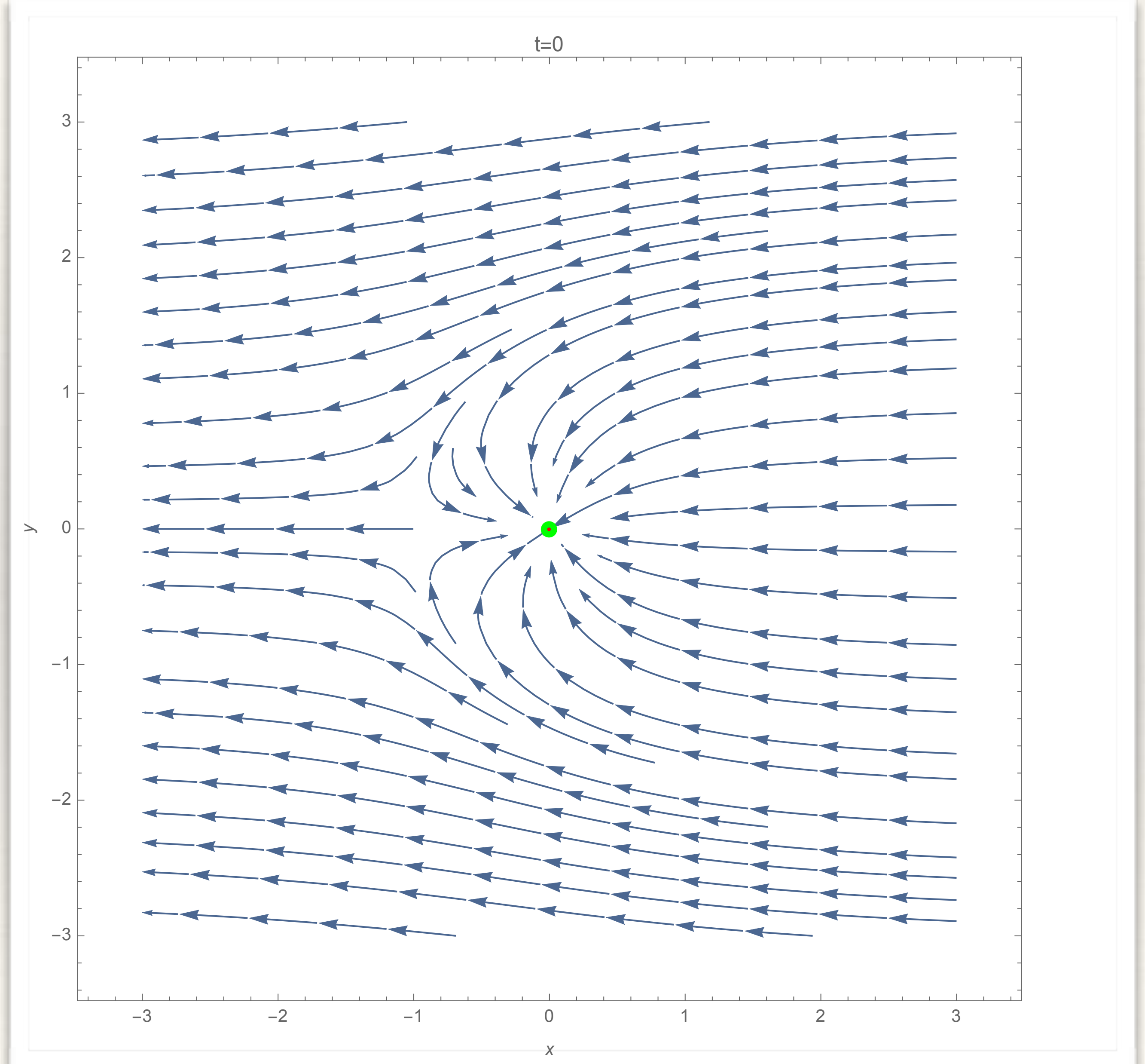
At distance Δx from a charge at $\vec{x}_0(t)$

$$\kappa(\vec{x}) = \cancel{O(\frac{1}{\Delta x})} + \cancel{O(1)} + O(\Delta x) + \dots$$

$$\approx \frac{3}{q}\underbrace{|\vec{E}_{\text{ext}}(\vec{x}) \times (\vec{x} - \vec{x}_0(t))|}_{}$$

No singularity - even for the total field!

$\vec{E} = \vec{E}_{\text{self}} + \vec{E}_{\text{ext}}$ - Total Field Lines



The field lines curve

Analytical Field Lines Curvature

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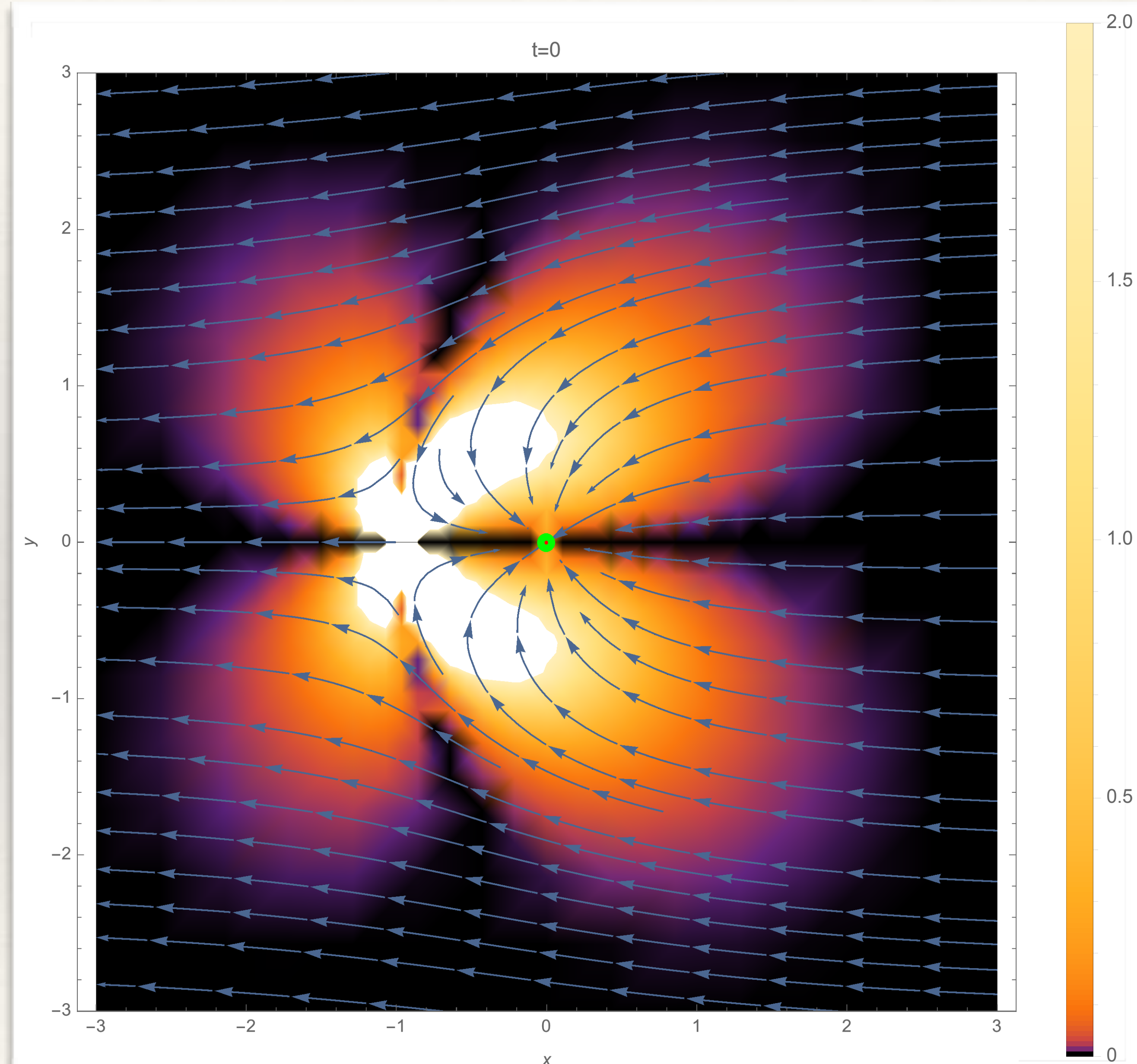
Total Field Lines Curvature (x-y plane)



The “Curvature Butterfly”

Analytical Field Lines Curvature

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At distance Δx from a charge at $\vec{x}_0(t)$

$$\kappa(\vec{x}) = \frac{3}{q}|\vec{E}_{\text{ext}}(\vec{x}) \times (\vec{x} - \vec{x}_0(t))| + O\left((\Delta x)^2\right)$$

As time passes by, the curvature is

$$0 = \kappa(\vec{x}_0(t + \Delta t)) \approx \frac{3(\Delta t)^2}{2q}|\vec{E}_{\text{ext}} \times \vec{a_0}(t)|$$

$\implies$ The charge accelerates in the direction of straight field-lines.

The magnitude of the acceleration is given by:

$$|\vec{a}| = c^2 \max(\kappa)$$

where the maximum is taken over a ball of radius

$$r = \frac{1}{3}\frac{e^2}{m_e c^2}$$

The classical electron radius

Acceleration magnitude is equal to the curvature in the direction perpendicular to the acceleration.

Action Principle & Consequences

Nearby the charge, the non-relativistic curvature vector:

$$\vec{\kappa} = \frac{3}{q}\vec{E}_{\text{ext}}(\vec{x}) \times (\vec{x} - \vec{x}_0(t))$$

Therefore (for no induction),

$$\nabla \times \vec{\kappa} = \frac{6}{q}\vec{E}_{\text{ext}}(\vec{x})$$

The scalar potential can be expressed using the curvature as

$$\phi = -\frac{q}{6}\int (\nabla \times \vec{\kappa}) \cdot d\vec{l}$$

Charges minimize the circulation of field lines curvature.

<u>Conclusions</u>

1. The electrostatic field lines around an accelerating charge curve

2. The charge accelerates along (locally) straight electrostatic field lines.

3. The electrostatic field lines curvature has no singularities, independently of the structure of the charge.

4. Charges travel along the path of least curvature circulation

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Conclusions

1. The ~~electrostatic~~ covariant electromagnetic field lines around an accelerating charge curve

2. The charge accelerates along (locally) straight ~~electrostatic~~ covariant electromagnetic field lines.

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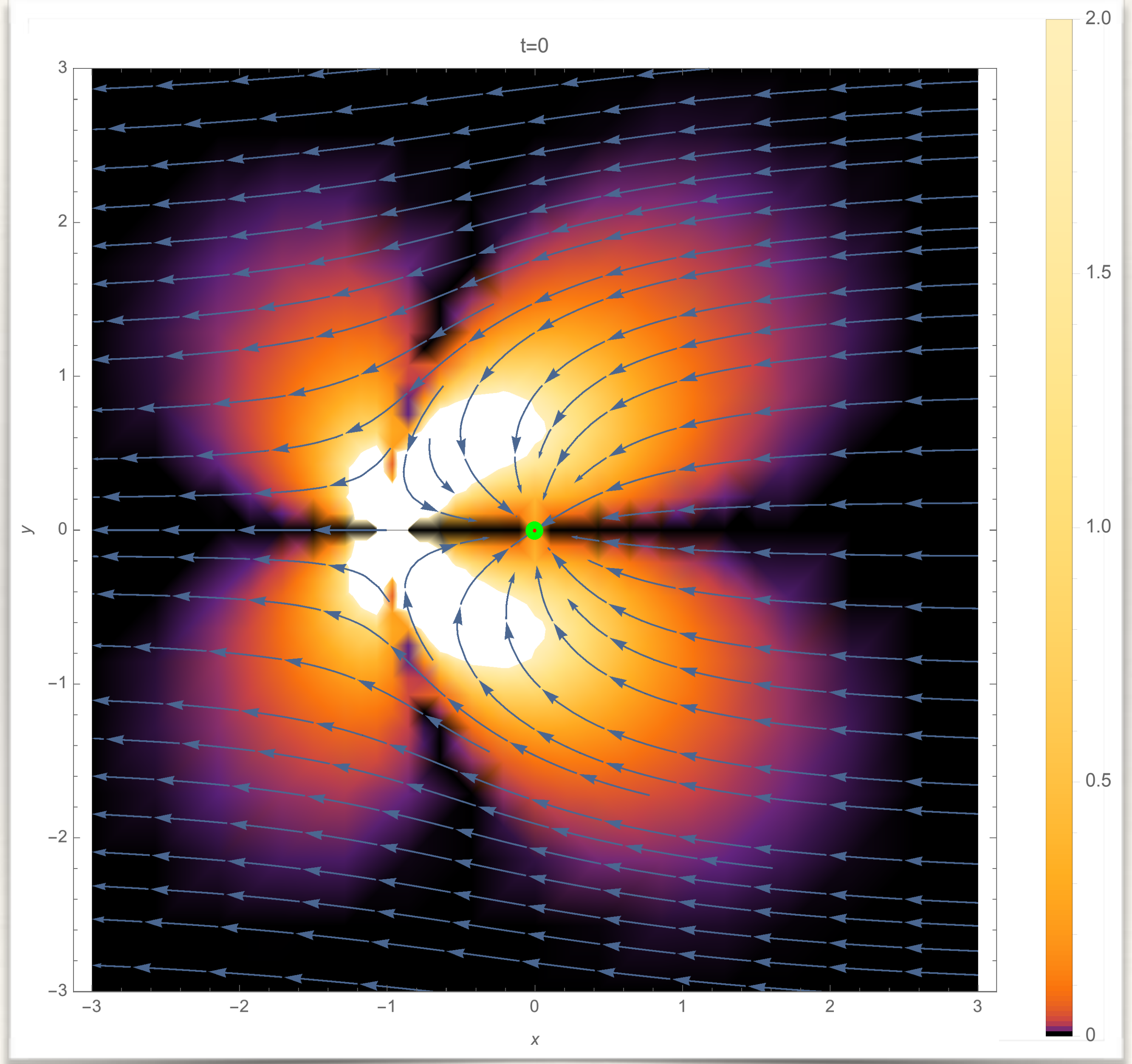
4. Charges travel along the path of least curvature circulation

Curvature of Relativistic Field Lines

- The electromagnetic field lines curvature is never singular (also for point charges)
- An electron travels along zero field lines curvature

Total Field Lines Curvature (x-y plane)



Curvature of Relativistic Field Lines

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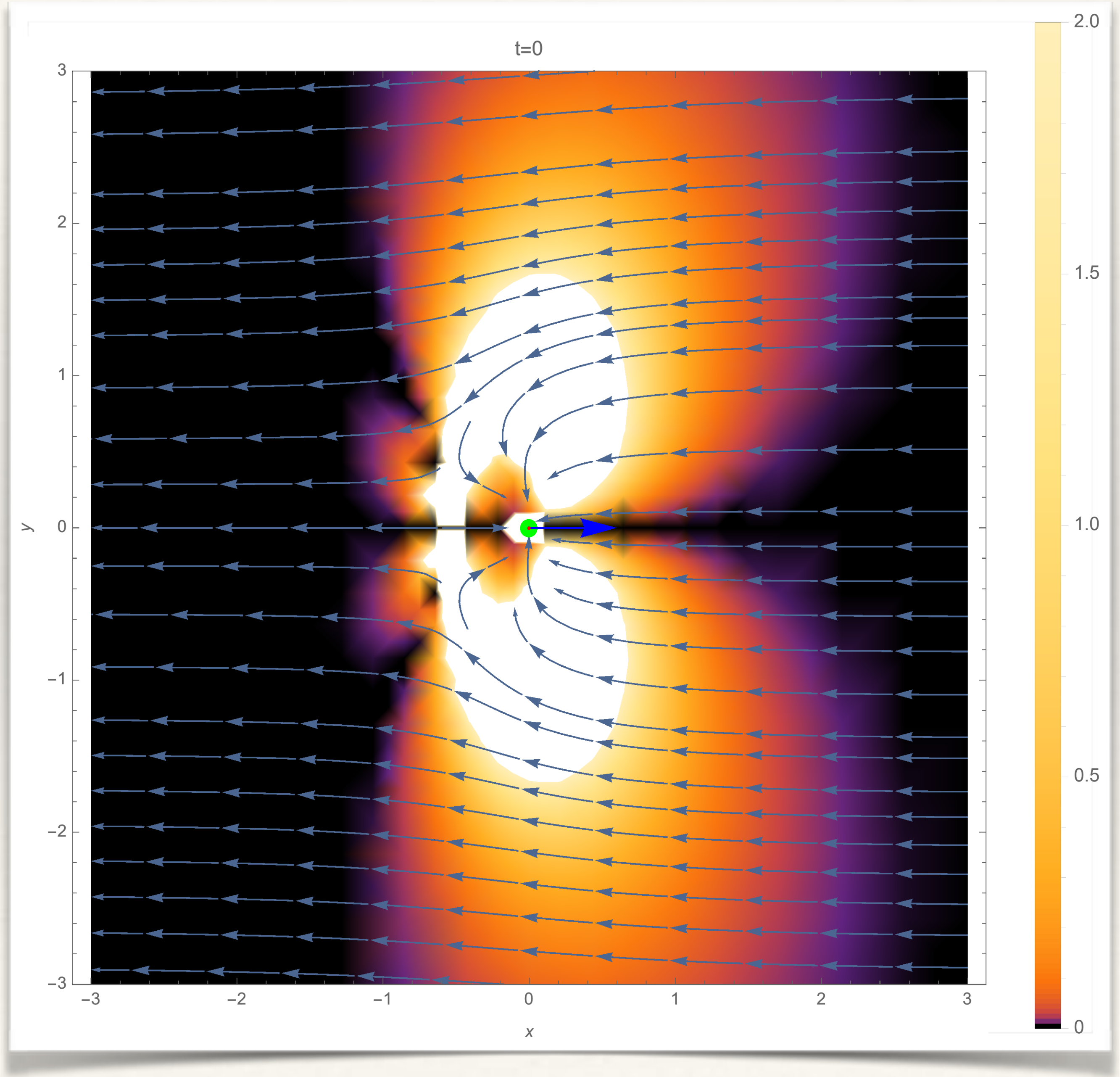
- An electron travels along zero field lines curvature
- Relativistic electrons have “squashed” curvature butterflies

Field Lines Curvature for parallel motion $v = 0.8$



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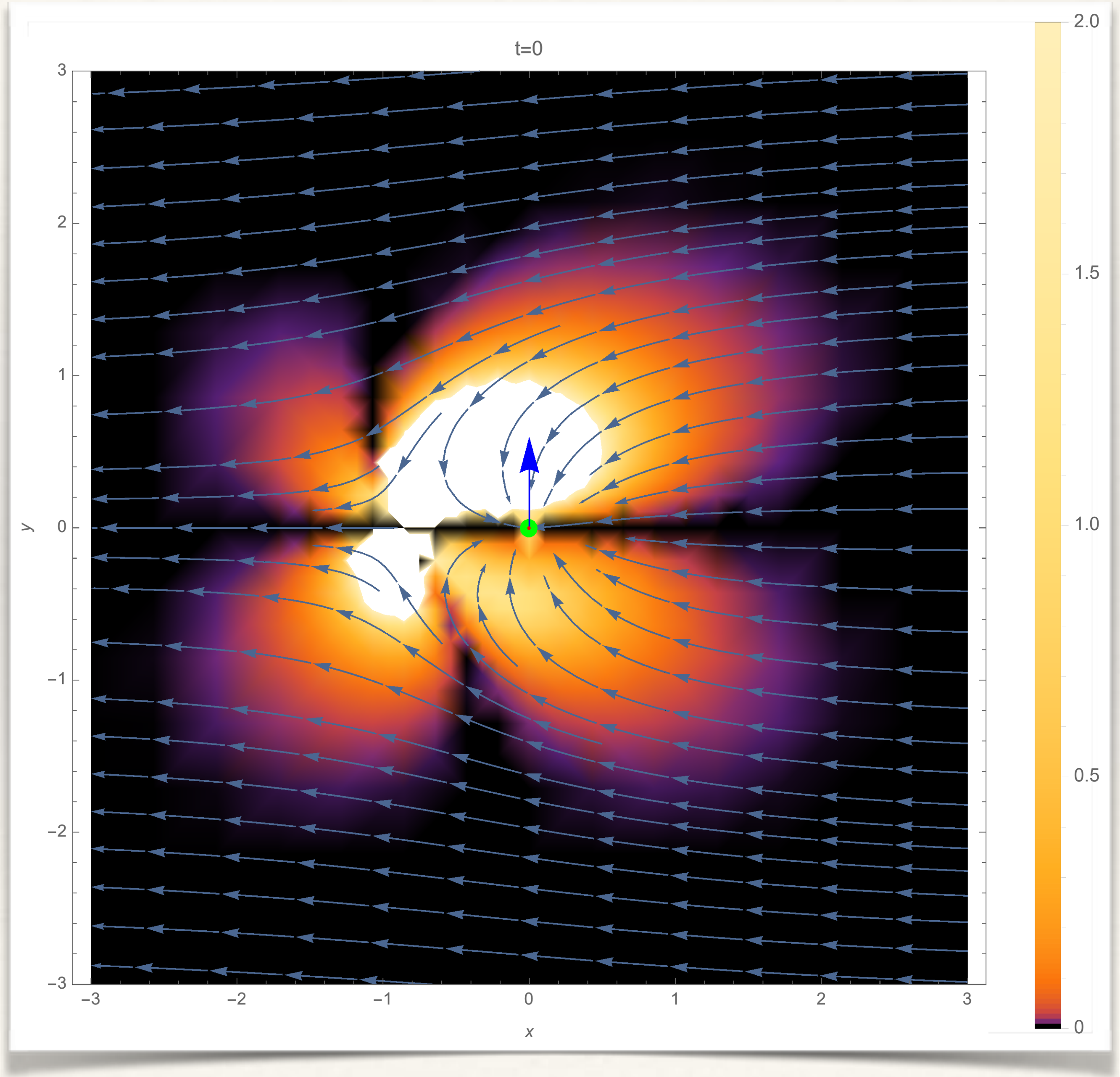
Field Lines Curvature for transverse motion $v = 0.6$



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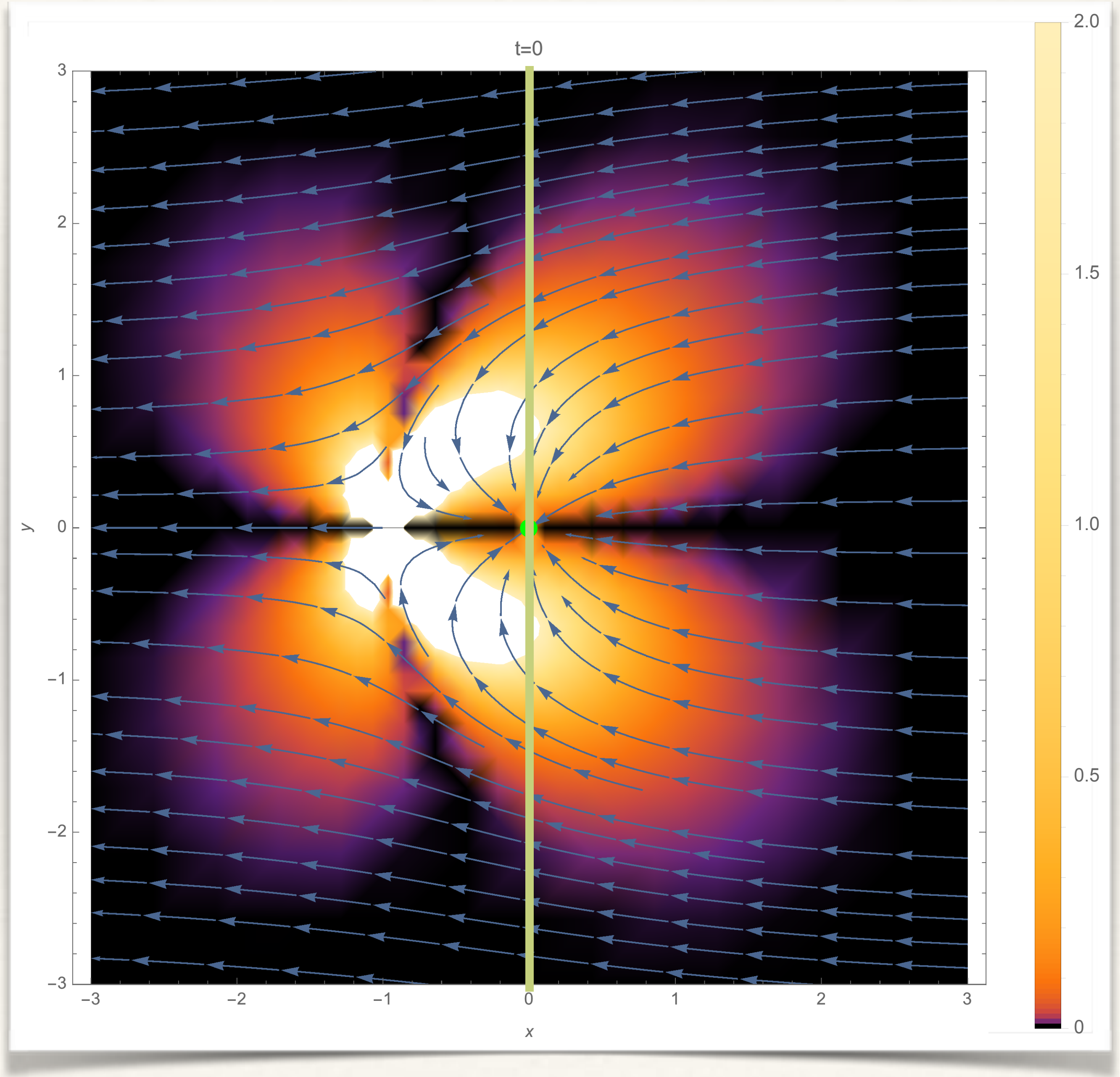

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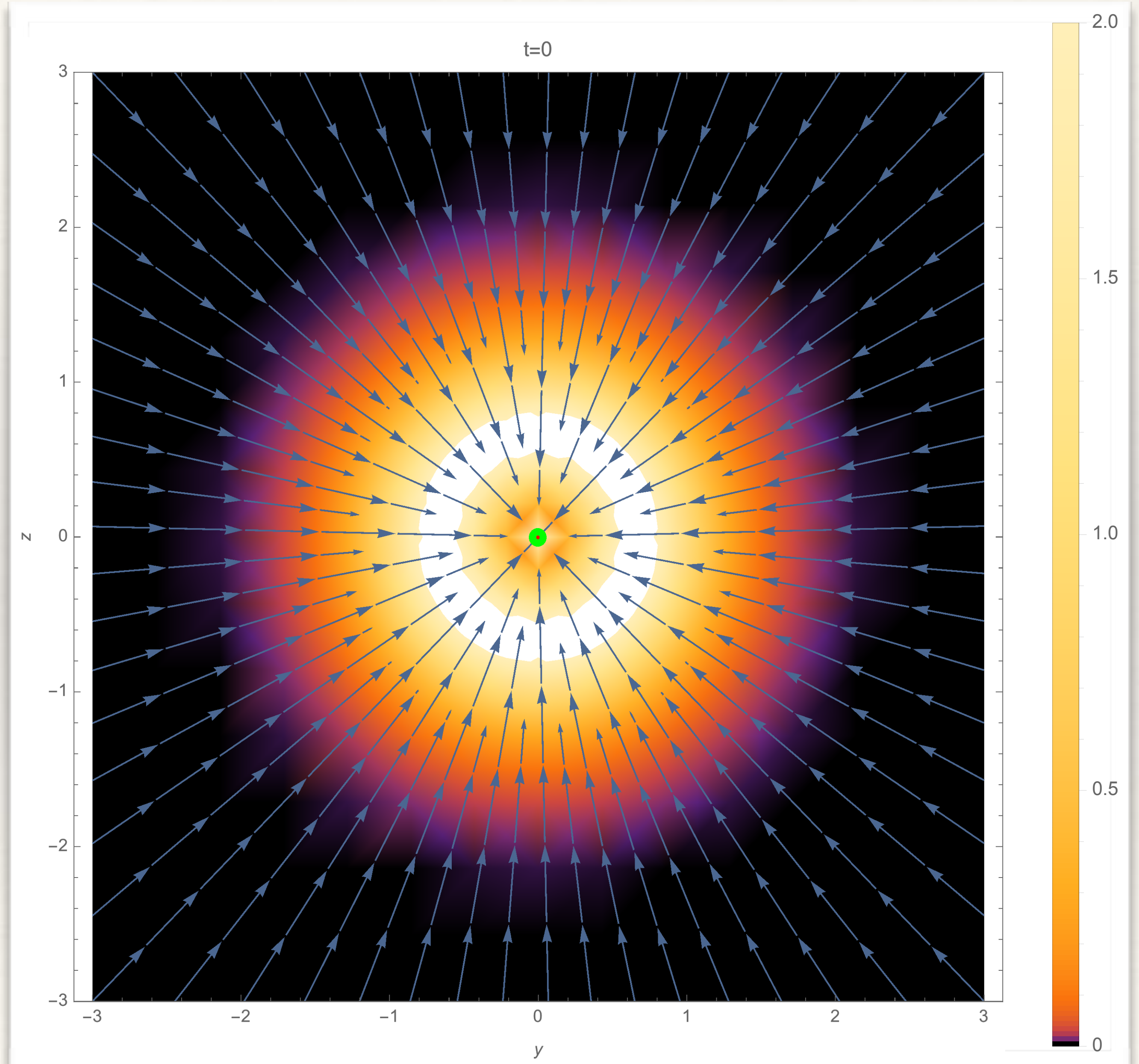
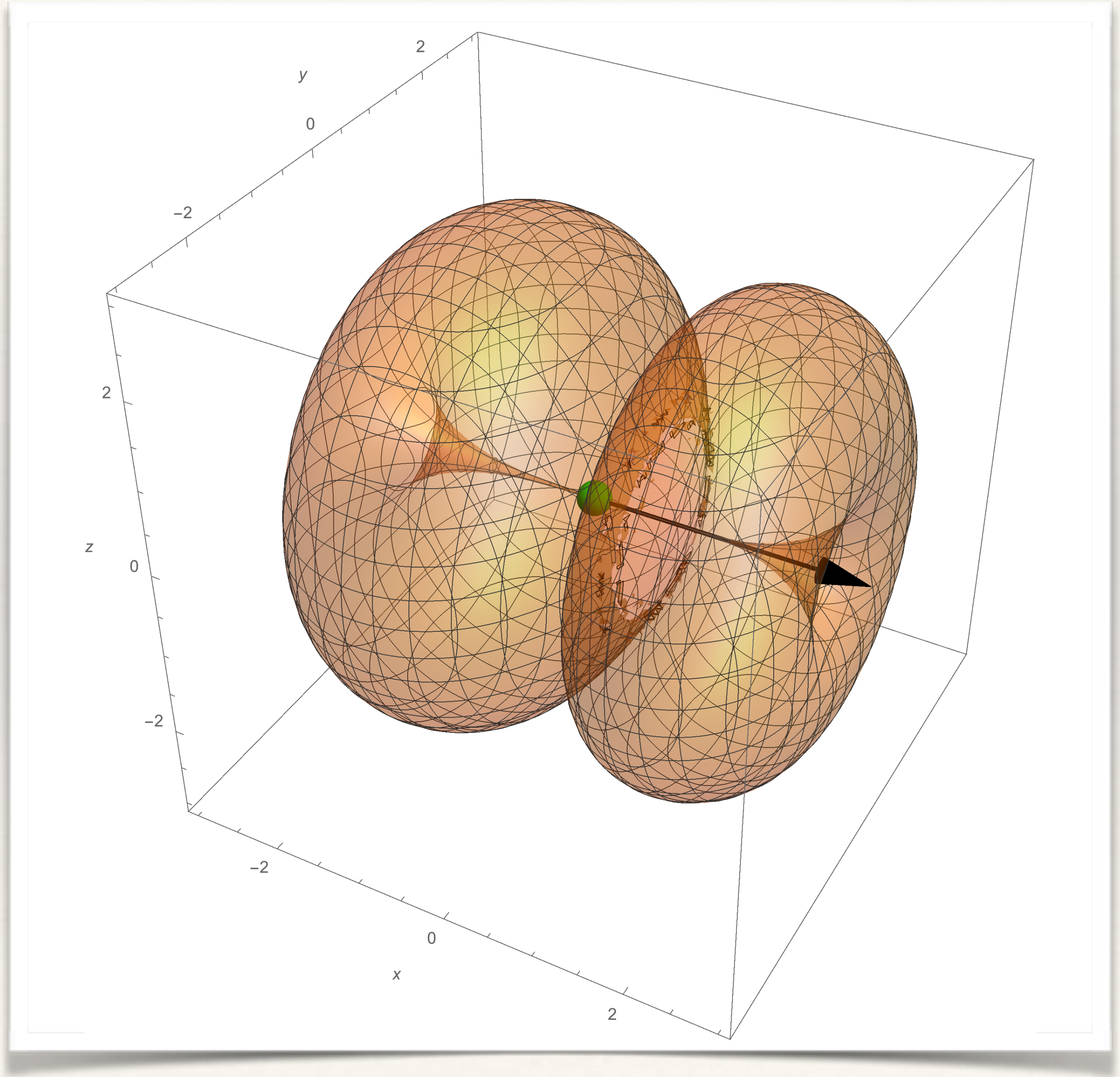
Total Field Lines Curvature (y-z plane)



Curvature of Relativistic Field Lines

- The electromagnetic field lines curvature is never singular (also for point charges)
- An electron ~~travels~~ accelerates along zero field lines curvature
- Relativistic electrons have "squashed" curvature butterflies
- The charge tears through a tiny curvature zero tube

Total Field Lines Curvature (x-y-z space)

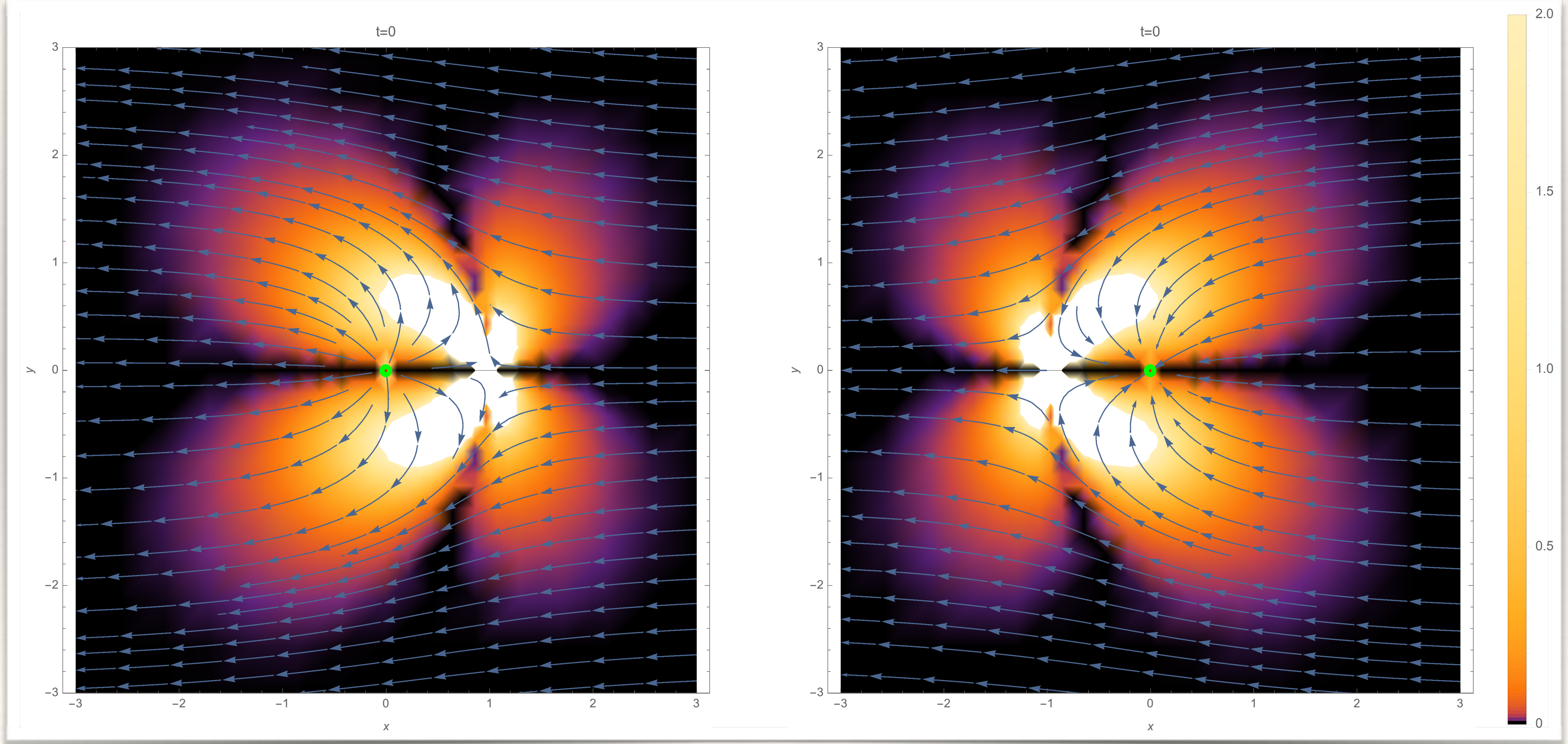


Positive vs. Negatives Charges

Example: Field Lines Curvature for Constant Electric Field in positive x-direction

Positron

Electron



Quote

"The propagation of light and therefore all radiant action occupies time; and a vibration of the line of force should account for the phenomena of radiation, so it is necessary that such vibration should occupy time also."

Michael Faraday

Quote

“The propagation of light and therefore all radiant action occupies time; and a vibration of the line of force should account for the phenomena of radiation, so <u>it is necessary that such vibration should occupy time</u> also.

I am not aware whether there are any data by which it has been, or could be ascertained, whether such <u>a power as gravitation</u> acts without occupying time or whatever lines of force being already in existence, such a lateral disturbance of them at one end… would require time, or must of necessity be felt at the other end.”

Michael Faraday

Covariant Field Lines Curvature

We wish to generalize the electrostatic field line eq: $$\frac{d\vec{\gamma}}{ds} = \vec{E}(\vec{\gamma}(s))$$

Covariant electromagnetic field line equation: $$\frac{d\gamma^\alpha}{ds} = f^\alpha(\gamma^\beta(s))$$

The curvature 4-vector: $$\kappa^\alpha = \frac{1}{f^2} f^\beta \frac{\partial f^\alpha}{\partial x^\beta} - \frac{1}{f^4}\left(f^\beta f^\gamma \frac{\partial f_\gamma}{\partial x^\beta}\right) f^\alpha$$

The squared scalar curvature: $$\kappa^2 = \kappa^\alpha \kappa_\alpha$$

Covariant field line tangent

$$f^\alpha := F^{\alpha\beta} u_\beta = F^{\alpha\beta}_{\text{self}} u_\beta + F^{\alpha\beta}_{\text{ext}} u_\beta$$

Liénard-Wiechert Self Field

The Liénard-Wiechert self field

$$F^{\alpha\beta}_{\text{self}} = \frac{q}{R^2}\left(U^\alpha k^\beta - U^\beta k^\alpha\right)$$

where

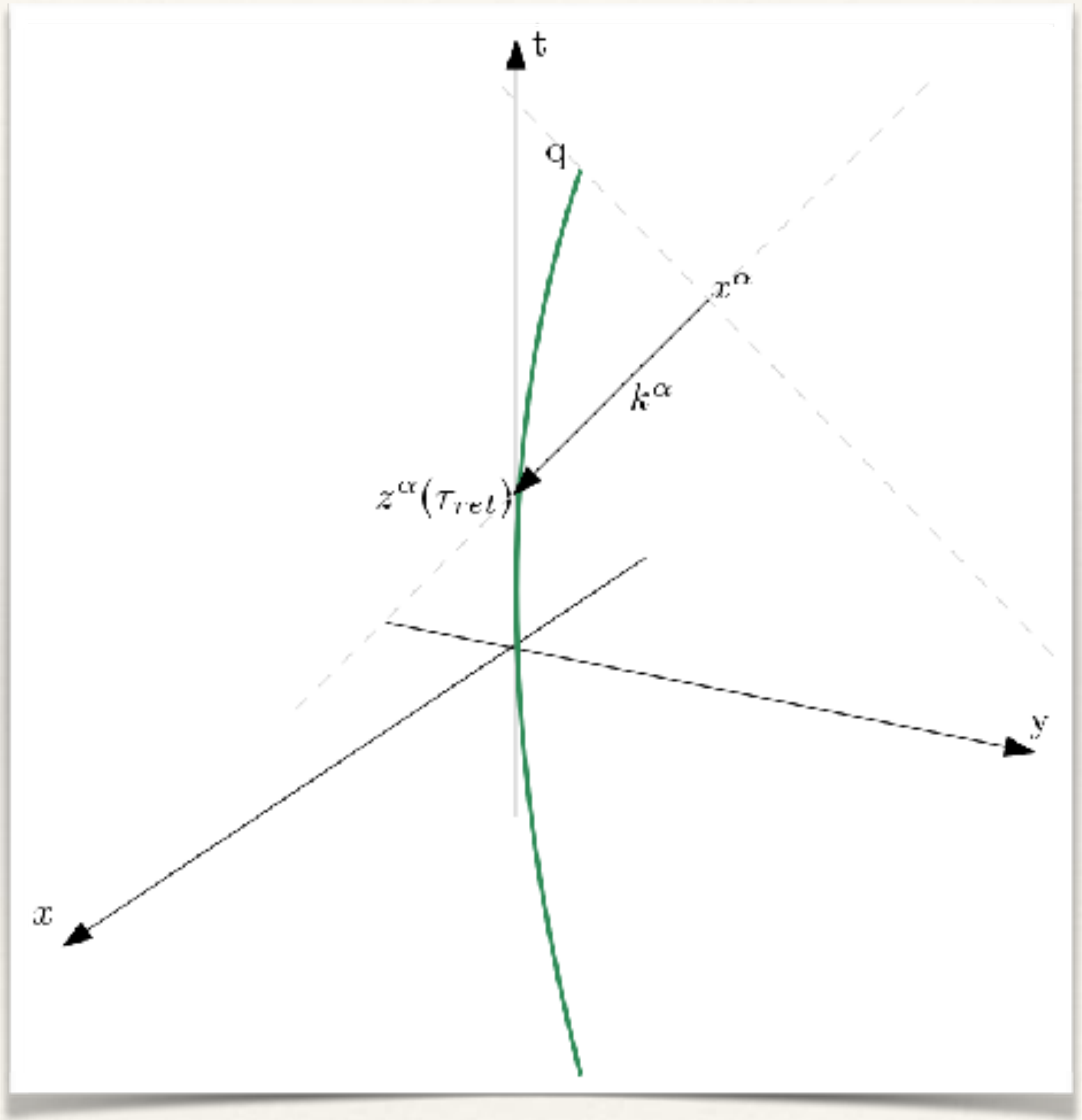
$$k^\alpha = x^\alpha - z^\alpha(\tau_{\text{ret}})$$ Null-vector

$$R = -k^\alpha u_\alpha$$ retarded distance

$$U^\alpha = B u^\alpha + a^\alpha$$ Synge vector

$$B = \frac{1-W}{R}$$ Plebanski invariant

$$W = -k^\alpha a_\alpha$$



Curvature = -Acceleration

To study the field lines curvature near the charge, take the limit

$$k^\alpha = x^\alpha - z^\alpha(\tau_{\text{ret}}) \longrightarrow 0$$

Let

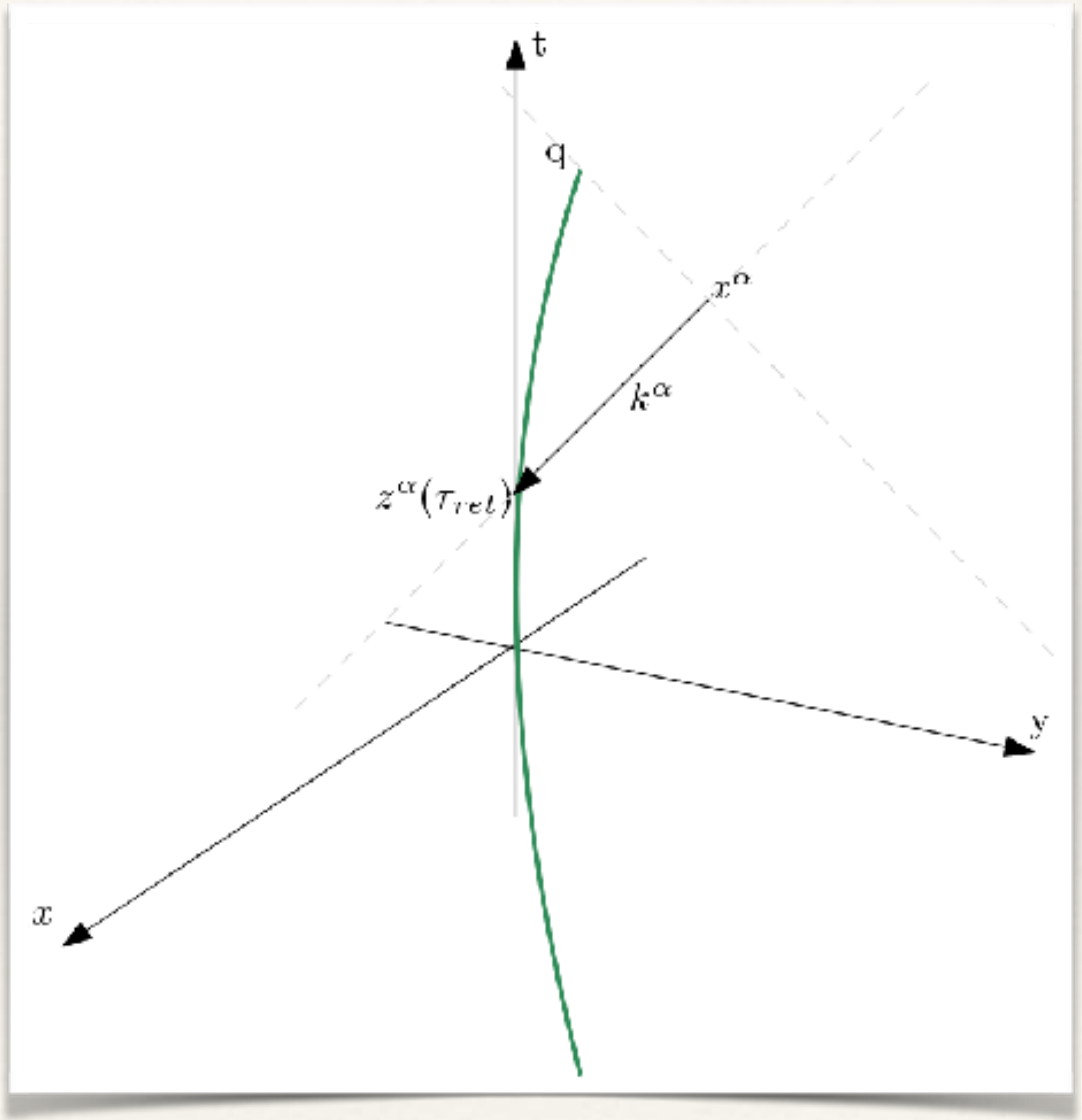
$$k^\alpha = (k^0, \vec{k}) = \underbrace{k^0}_{\varepsilon}\underbrace{(1, \hat{k})}_{\hat{k}^\alpha}, \quad \varepsilon \longrightarrow 0$$

And the field lines curvature is

$$\kappa^\alpha = -a^\alpha - \frac{\hat{W}}{\hat{R}^2}\hat{k}^\alpha + \frac{2\hat{W}}{\hat{R}}u^\alpha + O(\varepsilon)$$

where

$$\hat{R} = -\hat{k}^\alpha u_\alpha \quad \hat{W} = -\hat{k}^\alpha a_\alpha$$



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And the field lines curvature is

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where

$$\hat{R} = -\hat{k}^\alpha u_\alpha \qquad \hat{W} = -\hat{k}^\alpha a_\alpha$$

Therefore in the leading order

$$\kappa^\alpha = -a^\alpha + O(\varepsilon)$$

Whenever $\hat{W} = 0$

$$\Longleftrightarrow \hat{k}_\alpha \kappa^\alpha = 0$$

$$\Longleftrightarrow div\,(u^\alpha(\tau_{\text{ret}})) = 0$$

$\Longleftrightarrow$ Direction of maximal radiation emission

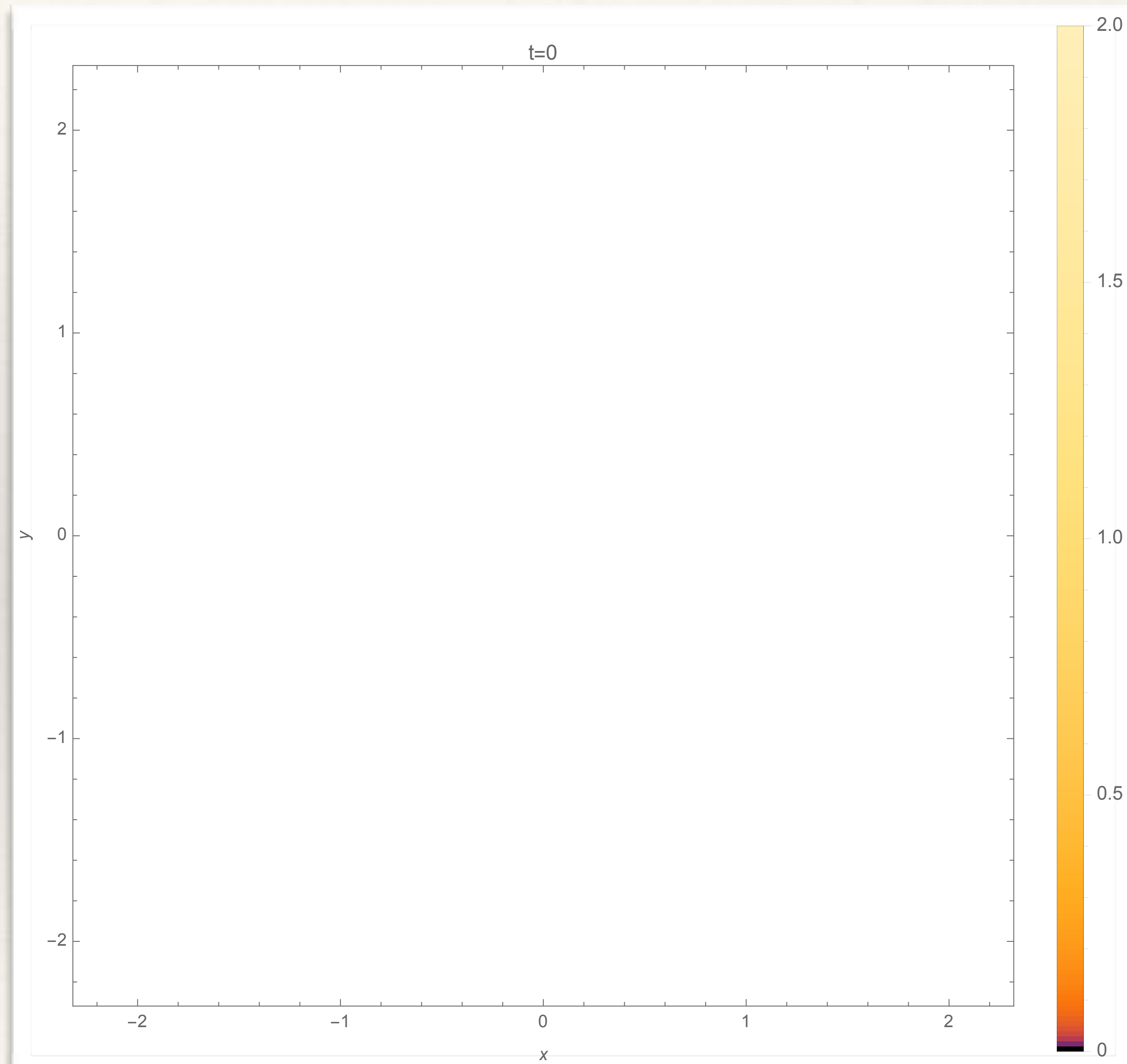
In the leading order,

covariant field lines curvature = (minus) charge acceleration

Curvature of Magnetic Field Lines

- An electron in constant magnetic field along the z-direction

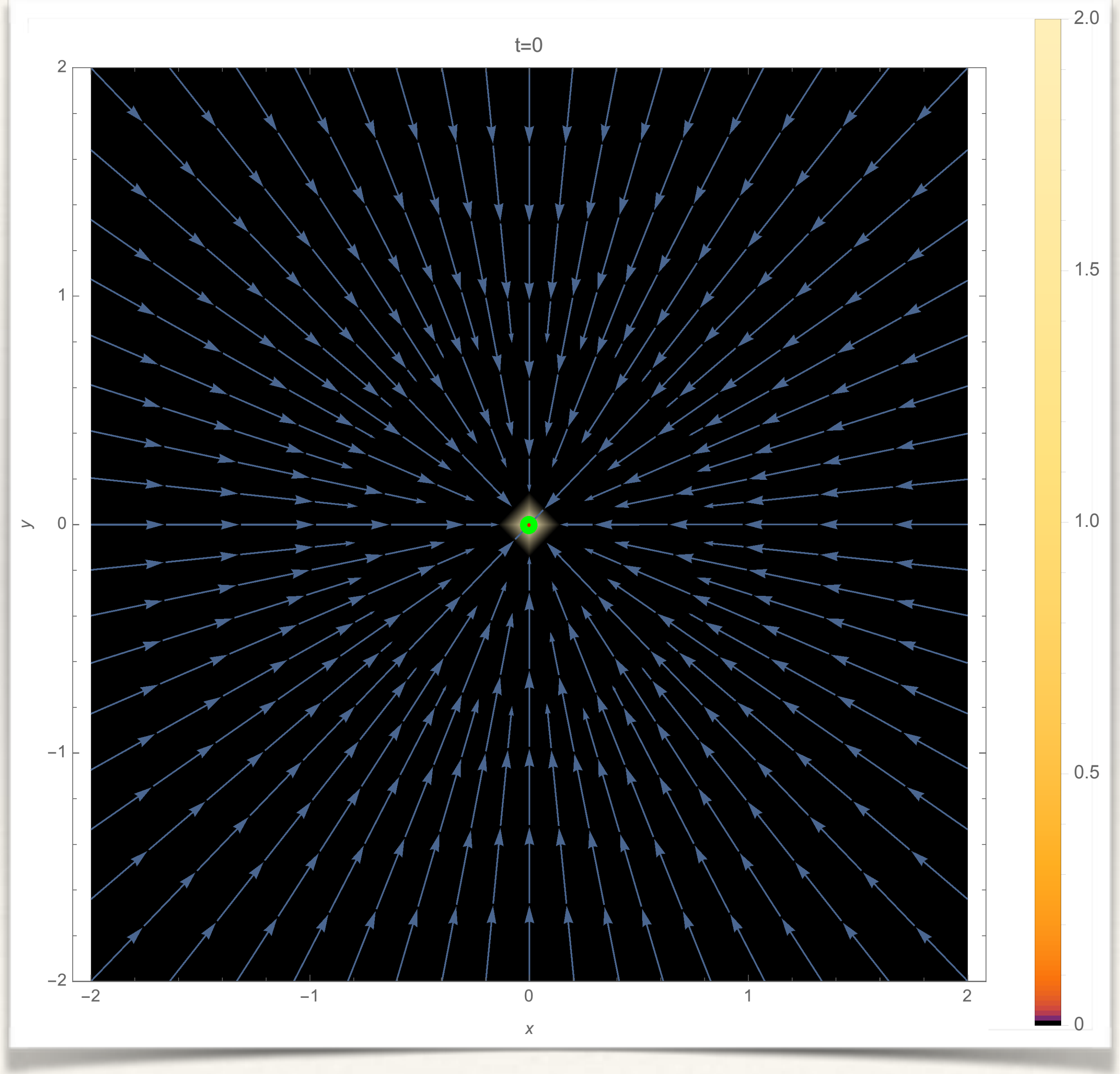
B (No Magnetic Field Lines in x-y Plane)



Curvature of Magnetic Field Lines

- An electron in constant magnetic field along the z-direction
- The total field lines point towards a resting electron

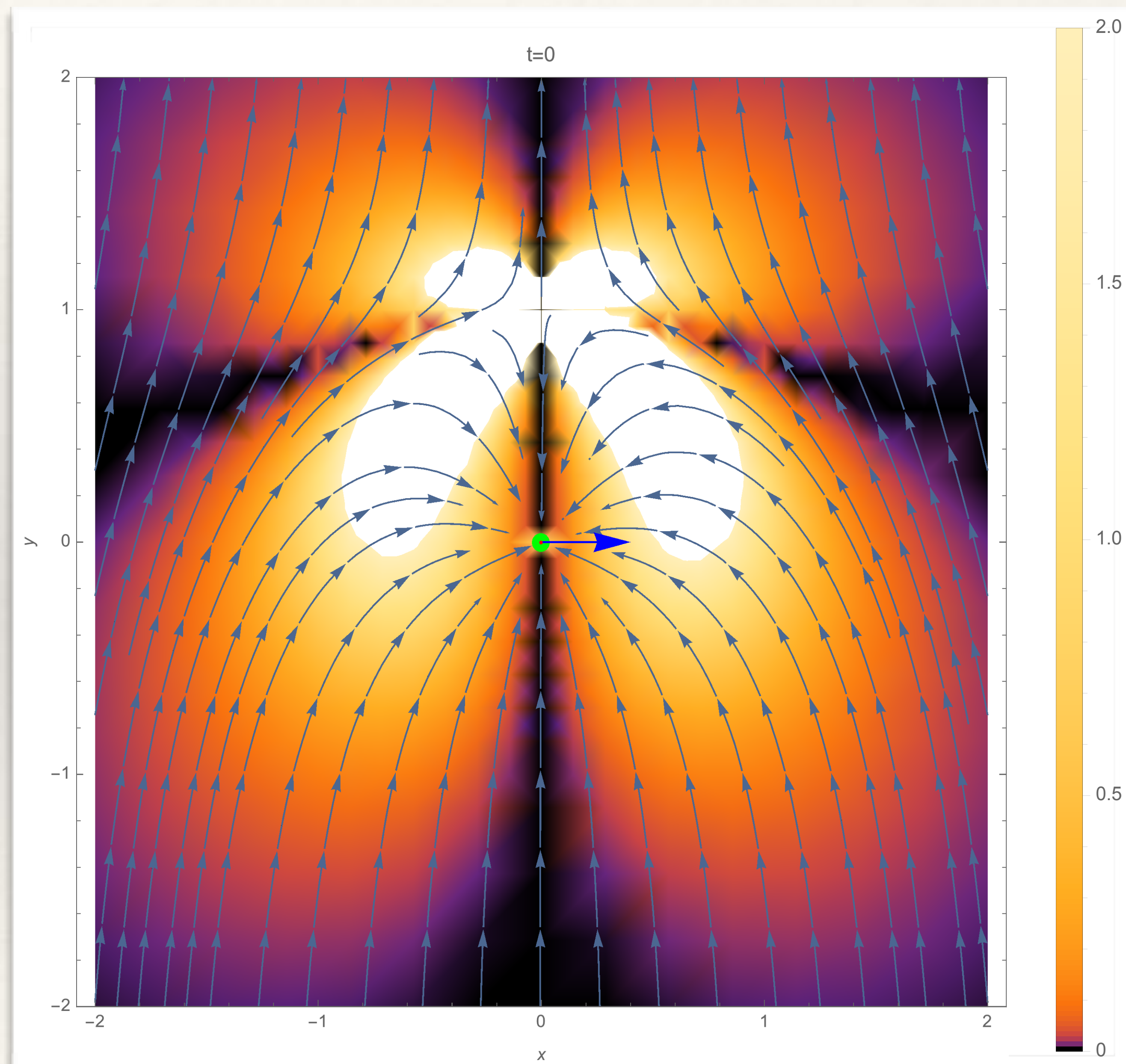
$$f^{\alpha} = F^{\alpha\beta} u_{\beta}$$ The Total Field Lines



Curvature of Magnetic Field Lines

- An electron in constant magnetic field along the z-direction
- The total field lines point towards a resting electron
- Electron's <u>motion</u> produces a <u>traverse</u> curvature butterfly as one would hope

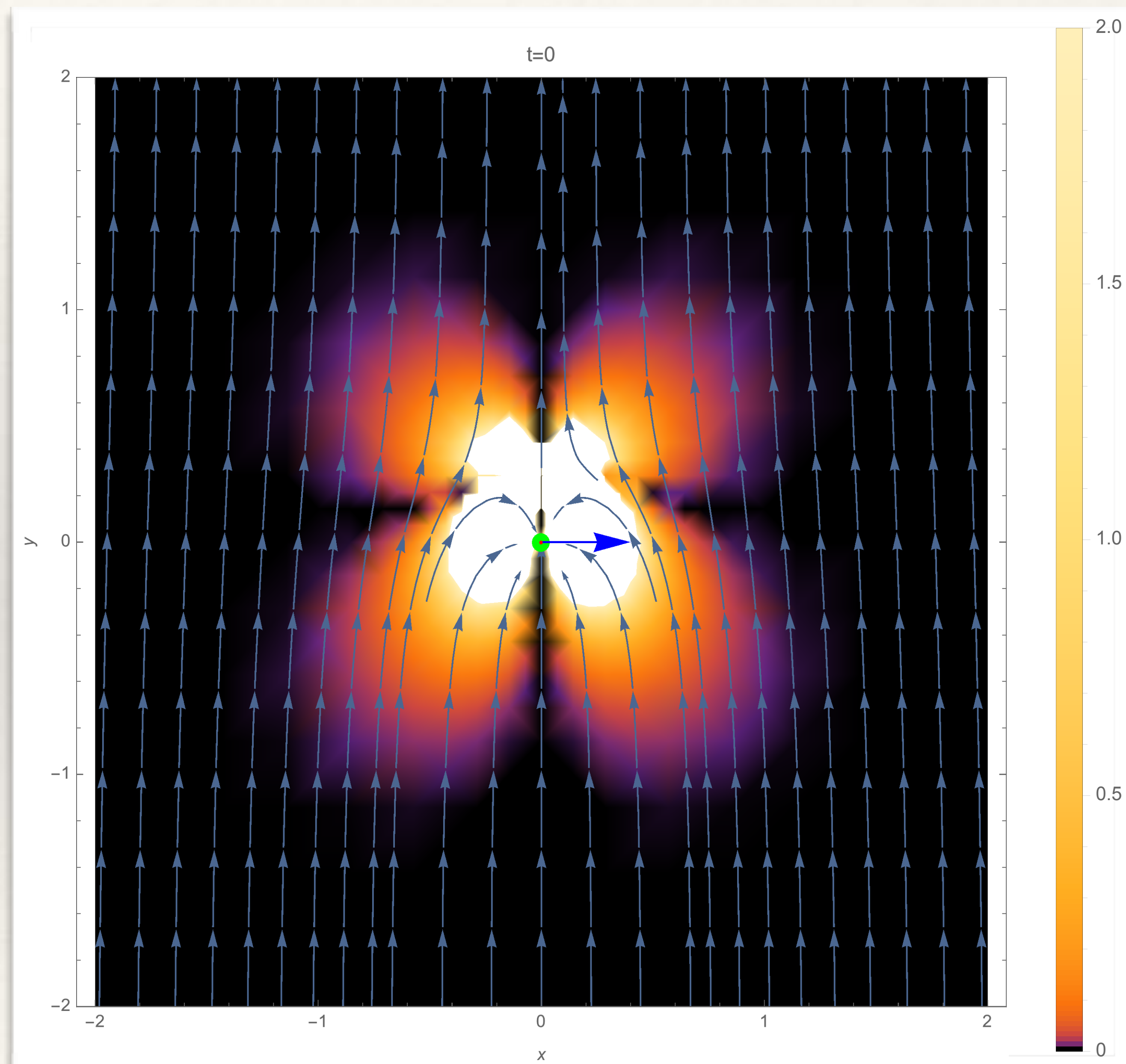
Electromagnetic Curvature for $v = 0.01$



Curvature of Magnetic Field Lines

- An electron in constant magnetic field along the z-direction
- The total field lines point towards a resting electron
- Electron's <u>motion</u> produces a <u>traverse</u> curvature butterfly as one would hope
- A magnetic curvature butterfly <u>shrinks</u> as v or B_{ext} increase

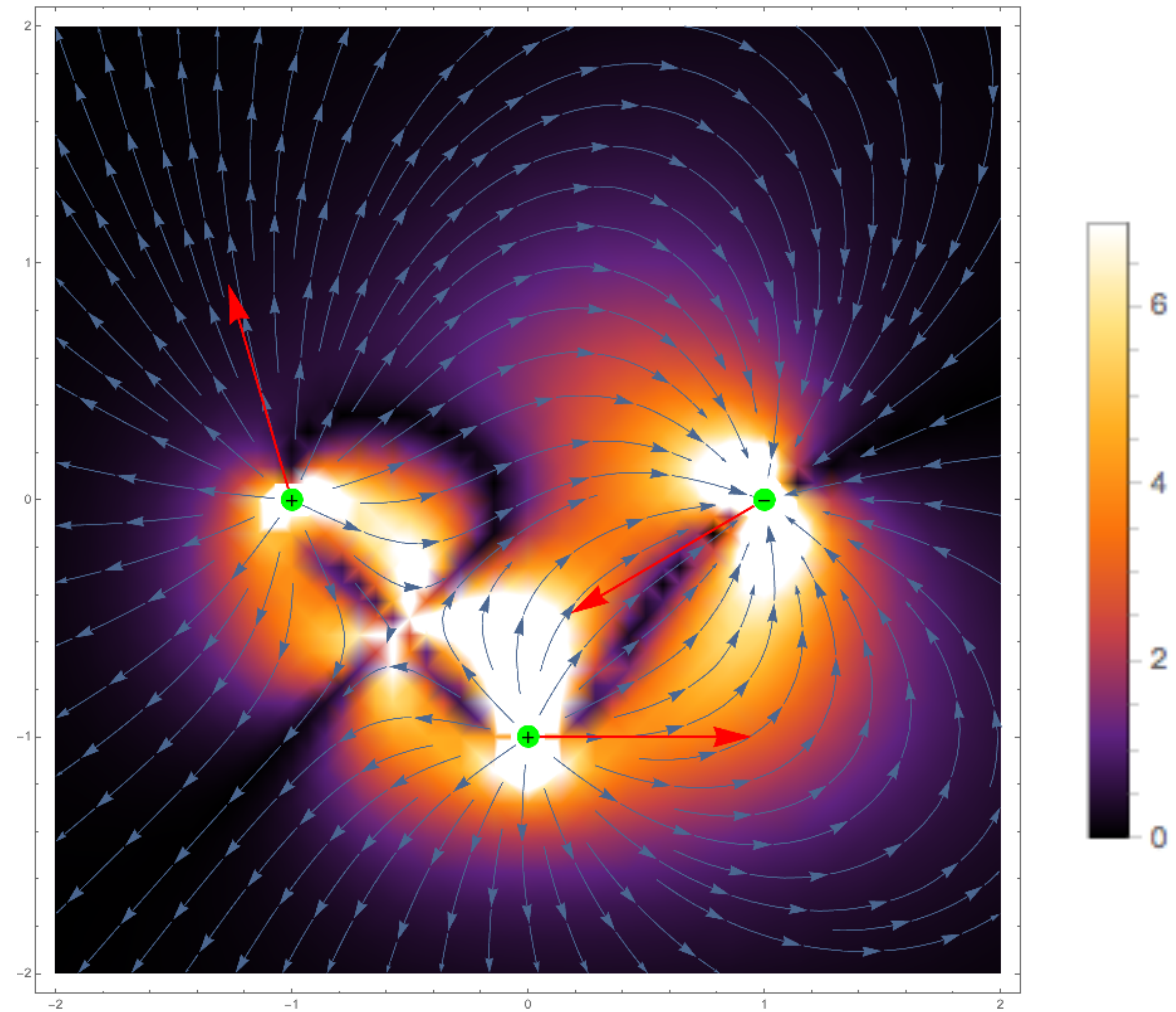
Electromagnetic Curvature for $v = 0.1$



Electromagnetic Curvature describes Multi-Charges Systems

Whenever there are multiple charges, each is accelerating along its locally flat electromagnetic field lines

Example: Electromagnetic Curvature for three charges



Correction to the Lorentz force?

Recall that when $\hat{W} = 0$ (acceleration perpendicular to null-vector)

$$\kappa^\alpha = -a^\alpha + O(\varepsilon)$$

A very long calculation gives the next order term:

$$\kappa^\alpha = -a^\alpha + \left[\left(-2a^2 - \frac{3}{q\hat{R}} \hat{k} \cdot f_{\text{ext}} + \frac{\hat{A}}{\hat{R}} \right) \hat{P}^\alpha + \hat{R}\dot{a}^\alpha + \frac{3\hat{R}}{q} f^\alpha_{\text{ext}} \right] \varepsilon + O(\varepsilon^2)$$

If we take $R = \varepsilon \hat{R} = \frac{1}{2}\tau_0$ with $\tau_0 = \frac{2}{3}\frac{e^2}{m} = 6.24 \times 10^{-24}\,\text{s}$

then the condition curvature=0 gives a relativistic equation of motion:

$$\underbrace{ma^\alpha = qF^{\alpha\beta}_{\text{ext}} u_\beta}_{\text{Lorentz force equation}} + \underbrace{m\tau_0 \left(\frac{1}{2}\dot{a}^\alpha - \frac{a^2}{\hat{R}} \hat{P}^\alpha + \frac{\hat{A}}{2\hat{R}^2} \hat{P}^\alpha \right)}_{\text{Radiation-reaction term. Meaning? Implications?}}$$

where

$$\hat{P}^\alpha = \hat{k}^\alpha - \hat{R} u^\alpha$$

$$\hat{A} = -\hat{k}^\alpha \dot{a}_\alpha$$

Summary

Current density

J^α

Maxwell's equations

$$\partial_\alpha F^{\alpha\beta} = 4\pi J^\beta$$

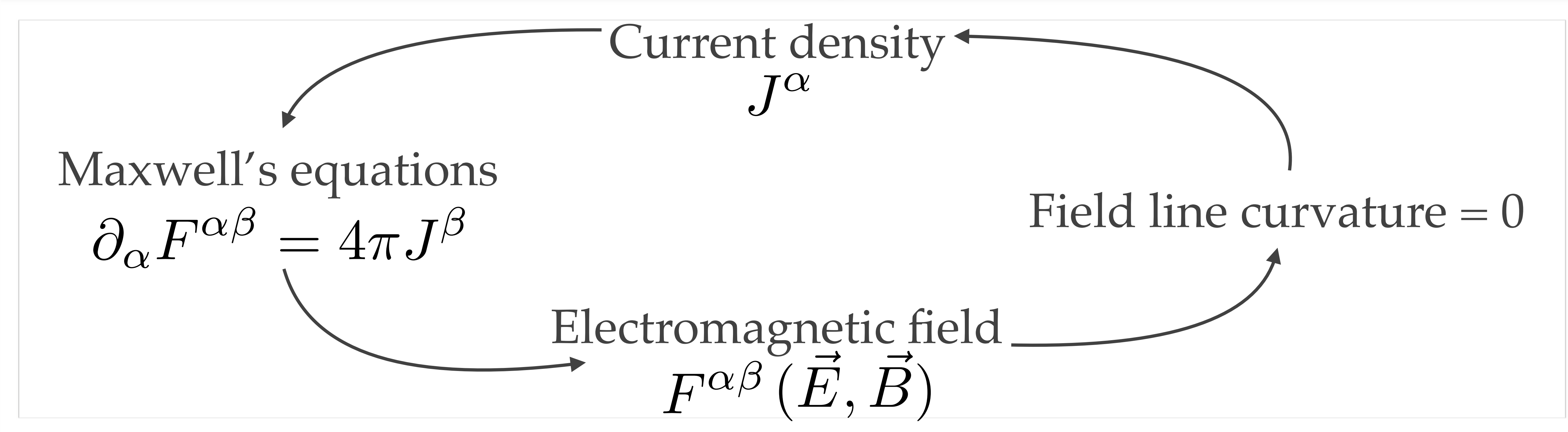
~~Lorentz Force equation~~

$$\cancel{m\dot{u}^\alpha = F^{\alpha\beta}_{\text{ext}} u_\beta}$$

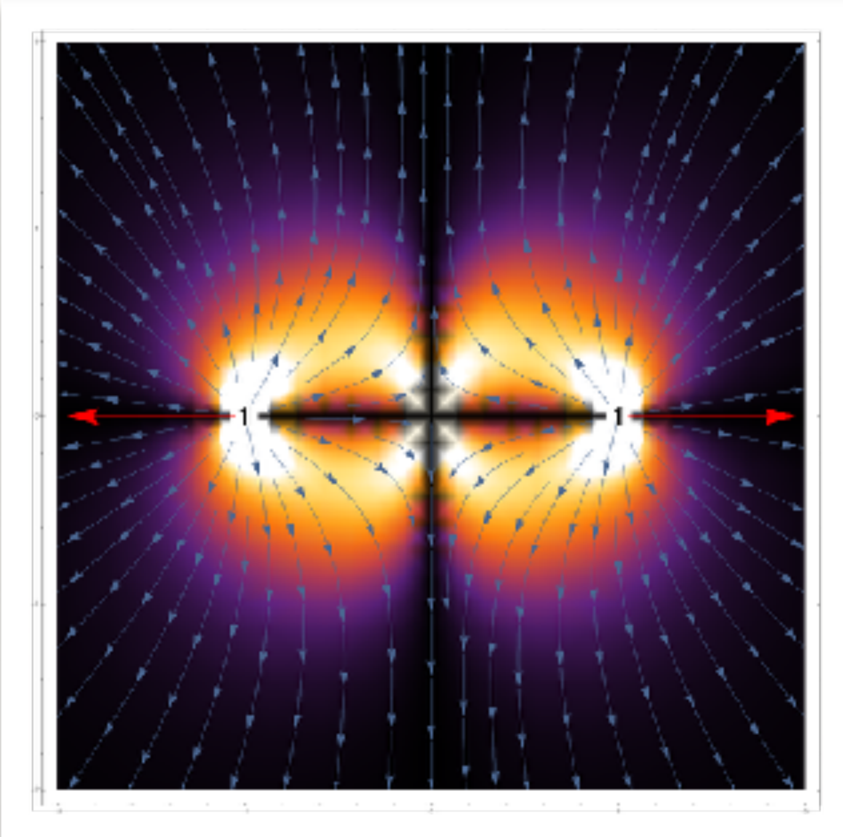
Electromagnetic field

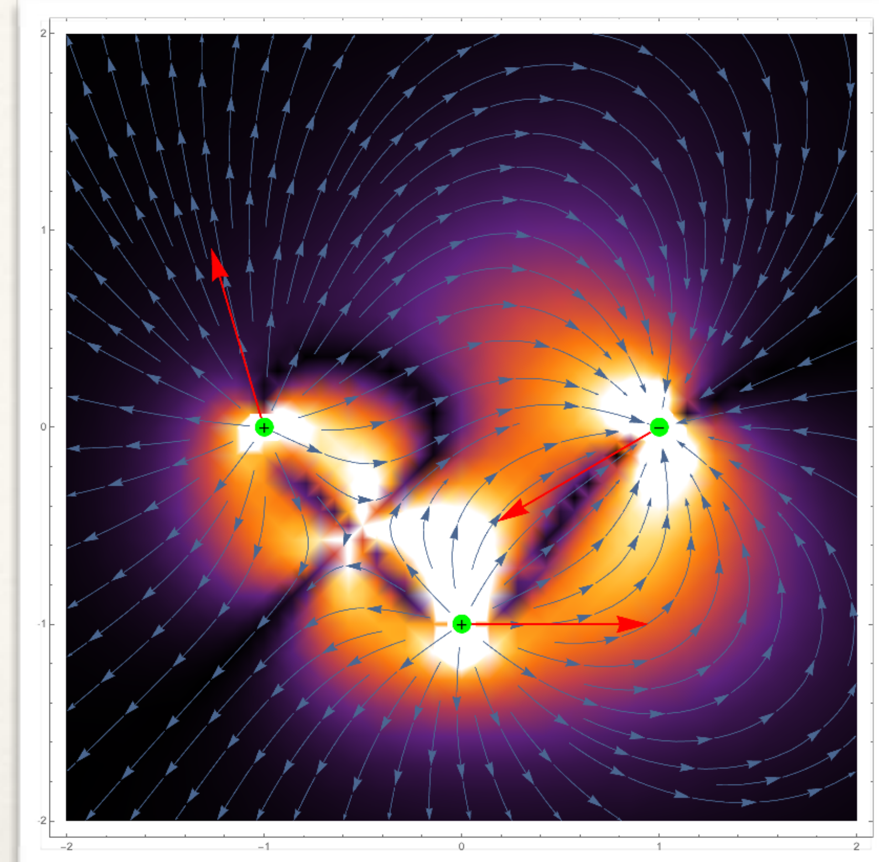
$$F^{\alpha\beta}(\vec{E}, \vec{B})$$

Summary



- The field lines curvature is always non-singular, also for point charges
- Accelerating charges curve the field lines, charges accelerate along locally straight spatial field lines. These are paths of least curvature circulation.
- At the charge, the covariant electromagnetic field lines curvature is minus the charge 4-acceleration





Thank you!

More info at:

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