THE “BIMBO” THEOREM

YARON HADAD, YOHAI MAAYAN

Theorem 1. *Let $f(z)$ be an analytic function at z_0. If the equation $f(z) = \omega_0$ (w.r.t z) has a zero of order m at $z = z_0$, then there exist $\varepsilon > 0, \delta > 0$ s.t. for all $\omega \in B(\omega_0, \varepsilon) \setminus \{\omega_0\}$ the equation $f(z) = \omega$ has precisely m simple zeros in $B(z, \delta)$.*

Proof. Let $F(z) = f(z) - \omega_0$. Since $F(z)$ is analytic, the zeros of F, F' are isolated. This implies that there exists $\varepsilon > 0$ such that in $\bar{B}(z_0, \varepsilon)$ the only zero of F and F' is z_0. Define

$$\delta = \min_t \left|F\left(z_0 + \varepsilon e^{it}\right)\right|$$

notice that clearly $\delta > 0$, since F is continuous and non-zero on the boundary of the ball $B(z_0, \varepsilon)$. Take $\omega \in B(\omega_0, \delta) \setminus \{\omega_0\}$, and denote

$$\omega = \omega_0 + \omega'$$

we have

$$0 < |\omega'| < \delta$$

therefore for all $z = z_0 + \varepsilon e^{it}$,

$$|F(z) - (F(z) - \omega')| = |\omega'| < \delta \leq F(z)$$

in other words, $F(z)$ and $F(z) - \omega'$ sastify Rouche’s theorem, therefore the number of zeros of F in $B(z_0, \varepsilon)$ is equal to the number of zeros of $F - \omega'$ in $B(z_0, \varepsilon)$, and both are equal to m. This implies that the number of zeros of $f(z) - \omega = F(z) - \omega'$ in $B(z_0, \varepsilon)$ is m as well. However, since $F' \neq 0$ there, each of this zeros is a simple zero as we wanted. Namely $F - \omega$ is a m-to-one function in $B(z_0, \varepsilon)$. □

Corollary 2. *If $f(z)$ is analytic and locally one-to-one at $z = z_0$ then $f'(z_0) \neq 0$.*

Proof. Assume by contradiction that $f'(z_0) = 0$, and denote $\omega_0 = f(z_0)$. The equation $f(z) = \omega_0$ has a zero of order ≥ 2 at $z = z_0$. By the previous theorem, $f(z)$ is at least two-to-one in a neighborhood of z_0. This contradicts the locally one-to-one. □

Theorem 3. *(Bimbo) Let $f(z)$ be an analytic function on a domain D. $f(z)$ maps boundaries to boundaries in D if and only if $f(z)$ is one-to-one and onto from D to $f(D)$.*

Proof. Homework... □