Complex functions and integral transformations
Session 5

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Möbius transformations

A Möbius transformation is a function $M : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ ($\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) of the form

$$M(z) = \frac{az + b}{cz + d} \tag{1}$$

such that $ad - bc \neq 0$.

The elementary Möbius transformations are

- Translation $z \mapsto z + b$
- Rotation and scaling $z \mapsto az$
- Inversion $z \mapsto \frac{1}{z}$

Any other Möbius transformation is a composition of the elementary Möbius transformations.

Their properties are:

- Every Möbius transformation is one-to-one, onto and conformal.
- $M(0) = \frac{b}{d}$, $M(-\frac{b}{a}) = 0$, $M(-\frac{d}{c}) = \infty$, $M(\infty) = \frac{a}{c}$
- The inverse Möbius transformation of (1) is $M^{-1}(w) = \frac{dw-b}{-cw+a}$.
- More generally, there is a correspondence between Möbius transformations and non-singular matrices (up to their determinant),

$$M(z) = \frac{az + b}{cz + d} \leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

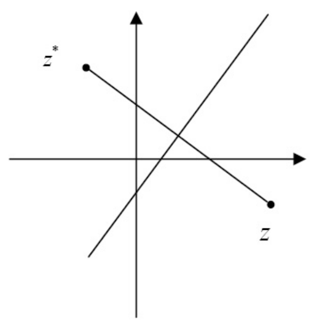
so that function composition is equivalent to Matrix multiplication and in particular,

$$M^-1(w) = \frac{dw - b}{-cw + a} \leftrightarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- A Möbius transformation turns a generalized circle (=circle or a line) to a generalized circle.
- A Möbius transformation is determined by its value on *three points.* So if $z_k \mapsto w_k$ for $k = 1, 2, 3$, it satisfies

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

- A Möbius transformation *preserves conjugate points.*



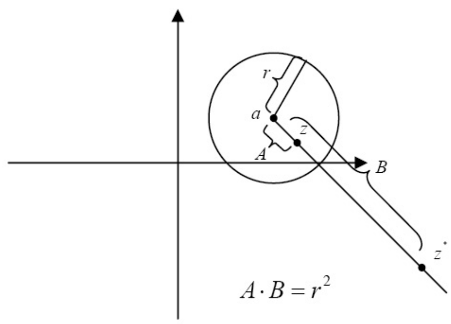
With respect to a line l, a conjugate point to a point $z \notin l$ is a point z^* on the other side of the line such that connecting the two points gives the perpendicular line to l, with its middle point being on the line l.

With respect to a circle C centered at a with radius r, the conjugate point z^* to z is on the same ray connecting the center of the circle a to z such that it satisfies

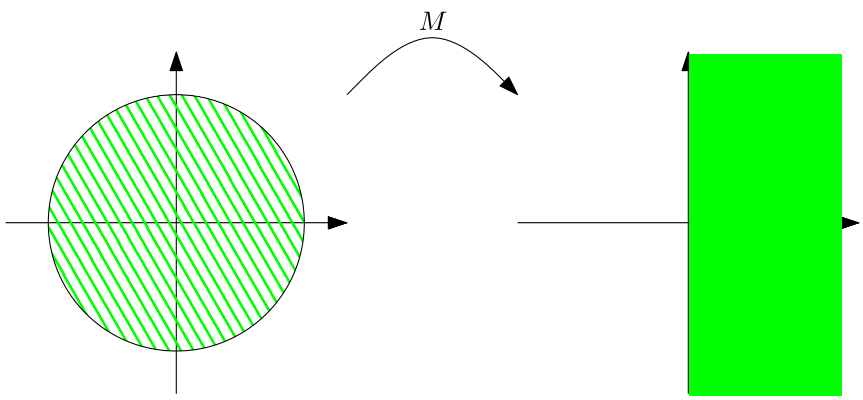
$$|z - a| \cdot |z^* - a| = r^2$$

A point on the circle is its own conjugate. The conjugate of the center of the circle is ∞.

Exercise 1 *Find the image of the unit ball by the transformation* $M(z) = \frac{1-z}{1+z}$. *What is the image of* $A = \{z||z| < 1, \Im z > 0\}$ *by this map?*



This is a Möbius transformation, so we just need to understand what happens to the boundary on three different points. Notice that $M(1) = 0, M(i) = -i, M(-1) = \infty$, so the unit circle transforms to the vertical line $\Re w = 0$. The origin maps to the point 1 and therefore the inside of the unit circle transforms to the right half plane.



Notice that on the real line $-1 \mapsto \infty, 0 \mapsto 1, 1 \mapsto 0$ so M maps the real lint to the real line. This with the knowledge that M is conformal gives that the set A is mapped to the lower-right quarter plane.

Exercise 2 *Find a Möbius transformation which takes the unit circle to the real line, such that $-1 \mapsto 1, i \mapsto 0, 1 \mapsto -1$. What is the image of the inside of the unit ball?*

Use the cross-ratio for Möbius transformations,

$$\frac{z-1}{z+1} \cdot \frac{i+1}{i-1} = \frac{w+1}{w-1} \cdot \frac{0-1}{0+1}$$

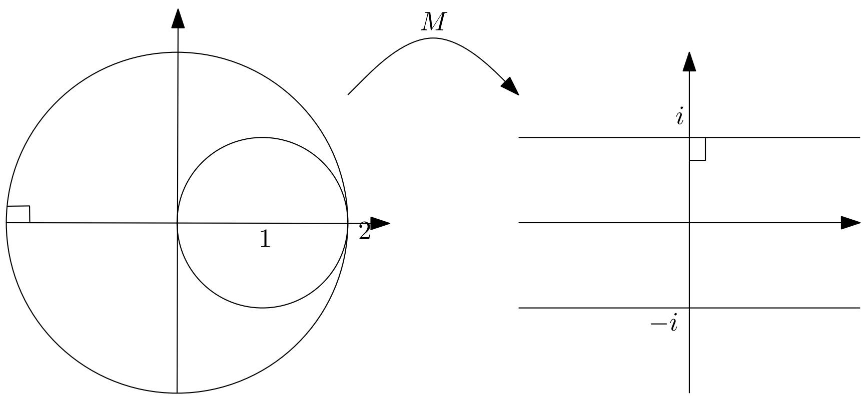
Inverting it gives

$$w = \frac{z-i}{-iz-1}$$

and it is easy to see that the unit ball transforms to the upper half-plane.

Exercise 3 *Find a Möbius transformation which takes* $D_1 = \{z||z| < 2, 1 < |z-1|\}$ *onto the strip* $D_2 = \{w|\Im w < 1\}$.

If a Möbius transformation and a circle is determined by three points, a line can be thought of as a circle that is passing through two points (on the line) together with the infinity (as a third point). In our case the only point in common between the two circles is 2 which must be mapped to ∞ (the point in common between the two lines).



Consider the Möbius transformation M that maps $2 \mapsto \infty, 0 \mapsto -i, -2 \mapsto i$. It must preserve angles, and in particular the 90 degree angle between the circle $|z| = 2$ and the real line must be transformed to a 90 degree angle between the line $w = i$ and the imaginary line. The cross-ratio is,

$$\frac{z-2}{z} \cdot \frac{-2}{-4} = \frac{w-\infty}{w+i} \cdot \frac{i+i}{i-\infty}$$

which is equivalent to

$$\frac{1}{2}(1 - \frac{2}{z}) = \frac{2i}{w+i}$$

Solving for w we get $w = M(z) = \frac{3iz+2i}{z-2}$.

Exercise 4 *Find a conformal map from* $D = \{z||z-1| > 1, |z-2| < 2\}$ *to the upper half-plane.*

We will do it in two stages. First we'll map D to the strip $S = \{w_1|0 < \Im w_1 < \pi\}$ and then we'll map this strip to the upper half-plane. For the first part, we may use a Möbius transformation similarly to the previous problem, so it maps $0 \mapsto \infty, 2 \mapsto 0, 4 \mapsto \pi i$, and therefore $M(z) = 2\pi i \frac{z-2}{z}$. For the second part, consider $g(w_1) = e_1^w$ which maps the strip to the upper half-plane. Therefore their composition solves the problem

$$f(z) = (g \circ M)(z) = e^{2\pi i \frac{z-2}{z}}$$

Exercise 5 *Find a Möbius transformation that maps the unit ball onto the ball* $\{w||w-1|<1\}$ *and maps* $0 \mapsto \frac{1}{2}, 1 \mapsto 0$.

We are already given two points that are fixed. A Möbius transformation is determined by three points, so we need to choose our third point. We will use the property that a Möbius transformation preserves conjugate points. With-respect-to the unit circle, the conjugate point to 0 is ∞. Since $0 \mapsto \frac{1}{2}$ then ∞ must be mapped to the conjugate point to $\frac{1}{2}$ w.r.t the circle $|w-1|=1$. Compute this conjugate point via the relation

$$|\left(\frac{1}{2}\right)^* - 1| \cdot |\frac{1}{2} - 1| = 1^2$$

So

$$|\left(\frac{1}{2}\right)^* - 1| = 2$$

and therefore $w(\infty) = \left(\frac{1}{2}\right)^* = -1$. Now we have three points $0 \mapsto \frac{1}{2}, 1 \mapsto 0, \infty \mapsto -1$ and the cross-ratio gives $w = \frac{-z+1}{z+2}$.

A riddle

Build four identical triangles from six identical rods. You are not allowed to modify the rods in anyway, and must use all of them.