# Calculus 2m1
# Session 4: Limits and Derivatives

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## Polar coordinates

Polar coordinates about $(x_0, y_0)$ are

$$x = x_0 + r\cos\theta \qquad y = y_0 + r\sin\theta$$

If $f(x,y)$ is defined about $(x_0, y_0)$ and satisfies

$$|f(x,y)| = |f(x_0 + r\cos\theta, y_0 + r\sin\theta)| \le g(r)$$

for every point about $(x_0, y_0)$ and $\lim_{r\to 0} g(r) = 0$ then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0$$

Intuitively, if the limit is independent of $\theta$ then $r \to 0$ means we are approaching the point $(x_0, y_0)$ independently of the direction.

Note that $\lim_{r\to 0} f(x_0+r\cos\theta, y_0+r\sin\theta) = 0$ *does not imply* that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0$. The limit must be bounded by a function that is independent of $\theta$.

**Exercise 1** *Find the limit* $\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)}$ *if it exists.*

Using polar coordinates about $(0,0)$,

$$\begin{aligned} \frac{\sqrt{x^2+y^2}}{\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)} &= \frac{r}{\arccos\left(\frac{r\cos\theta}{r}\right)} \qquad (1) \\ &= \frac{r}{\arccos(\cos\theta)} \\ &= \frac{r}{\theta} \text{ for } \theta \in [0,\pi] \end{aligned}$$

This means it does not satisfy the conditions of the theorem and there isn't a bound independent of $\theta$! If we take the family of spirals $\theta = kr$ (for $k \neq 0$) we see that

$$\begin{aligned} \frac{\sqrt{x^2+y^2}}{\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)} &= \lim_{r\to 0} \frac{r}{\theta} \\ &= \lim_{r\to 0} \frac{1}{k} \\ &= \frac{1}{k} \end{aligned} \tag{2}$$

which depends on $k$. Therefore the limit does not exist.

# Derivatives

**Exercise 2** *Let*

$$f(x,y) = \begin{cases} \frac{x \sin y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

1. *Where is $f$ continuous?*
2. *Compute $f_x$ on the entire plane.*
3. *Is $f_x$ continuous at $(0,0)$?*

1. As a composition of elementary functions, it is clear that $f$ is continuous for $(x,y) \neq (0,0)$. At the origin, let's use polar coordinates

$$\begin{aligned} |f(x,y)| &= |f(r\cos\theta, r\sin\theta)| \\ &= \left|\frac{r\cos(\theta)\sin(r^2\sin^2\theta)}{r^2}\right| \\ &\leq \left|\frac{r\cos(\theta) r^2 \sin^2\theta}{r^2}\right| \\ &= r|\cos\theta\sin^2\theta| \\ &\leq r \equiv g(r) \end{aligned} \tag{3}$$

   and $g(r) \to 0$ as $r \to 0$, therefore $f$ is continuous at the origin.

2. For all $(x,y) \neq (0,0)$ we may use rules of derivatives, and

$$\frac{\partial f}{\partial x} = \frac{\sin(y^2)(y^2 - x^2)}{(x^2+y^2)^2}$$

At the origin, we use the definition of the partial derivative,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0}{h} = 0$$

Therefore,

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{\sin(y^2)(y^2-x^2)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

3. There are two methods to do this. First, using polar coordinates

$$\begin{aligned} f_x &= \frac{\sin(r^2\sin^2\theta)r^2(\sin^2\theta - \cos^2\theta)}{r^4} \qquad (4) \\ &= \frac{\sin(r^2\sin^2\theta)}{r^2\sin^2\theta} \cdot \sin^2\theta(\sin^2\theta - \cos^2\theta) \\ &= 1 \cdot \sin^2\theta(-\cos 2\theta) \end{aligned}$$

The limit depends on $\theta$ and therefore does not exist (e.g. for $\theta = 0$ we get 0, and for $\theta = \pi/2$ we get $-1$).

An alternative method is to take paths of the form $(t, mt)$ and then

$$\begin{aligned} \lim_{t\to 0} f_x(t,mt) &= \lim_{t\to 0} \frac{\sin(m^2t^2)t^2(m^2-1)}{t^4(1+m^2)^2} \qquad (5) \\ &= \lim_{t\to 0} \frac{\sin(m^2t^2)}{m^2t^2} \cdot m^2t^2 \frac{(m^2-1)}{t^2(1+m^2)^2} \\ &= \frac{m^2(m^2-1)}{(1+m^2)^2} \end{aligned}$$

which depends on $m$. Therefore the limit doesn't exist and $f_x$ is discontinuous at the origin.

# Differentiability

**Definition 1** *We say $f(x,y)$ is differentiable at $p_0 = (x_0, y_0)$ if*

$$f(x,y) = f(x_0,y_0) + A(x-x_0) + B(y-y_0) + \varepsilon(x,y)\sqrt{(x-x_0)^2 + (y-y_0)^2}$$

*where $A, B$ are constants and*

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon(x,y) = 0$$

**Theorem 2** *If $f$ is differentiable at $(x_0, y_0)$ then:*

- *$f$ is continuous at $(x_0, y_0)$.*
- *$A = f_x(x_0, y_0)$ and $B = f_y(x_0, y_0)$ (and in particular the partial derivatives exist).*

Geometrically, this means we can approximate the graph of $f(x, y)$ with a tangent plant at $p_0 = (x_0, y_0)$. If we denote $z = f(x, y)$ and $z_0 = f(x_0, y_0)$ and neglect the $\varepsilon$ term then we get

$$-f_x(p_0)x - f_y(p_0)y + z = z_0 - f_x(p_0)x_0 - f_y(p_0)y_0$$

and the right-hand side is indeed a constant. This is the equation of a tangent plane with normal $(-f_x(p_0), -f_y(p_0), 1)$.

How do we check if $f$ is differentiable at $p_0 = (x_0, y_0)$?

1. Check continuity, not continuous $\Longrightarrow$ not differentiable.
2. Check existence of partial derivatives, don't exist $\Longrightarrow$ not differentiable.
3. Plug into the formula, solve for $\varepsilon(x, y)$ and check if $\varepsilon \to 0$. If it does, the function is differentiable, otherwise it is not.

**Definition 3** *The gradient of $f$ at $p_0$ is the vector*

$$grad f(p_0) = \nabla f(p_0) = (f_x(p_0), f_y(p_0))$$

*The gradient gives the direction of highest slope of the function $f$.*

**Exercise 3** *Let*

$$f(x, y) = \begin{cases} \frac{xy^3+y^4}{(x^2+y^2)^k} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

*Check if $f$ is differentiable at the origin for $k = 1, 1.5, 2$.*

We start by checking continuity using polar coordinates,

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= |\frac{r^4(\cos\theta\sin^3\theta + \sin^4\theta)}{r^2k}| \\ \leq 2r^{4-2k} \equiv g(r) \end{aligned} \tag{6}$$

Now $\lim_{r\to 0} g(r) = 0$ if $4 - 2k > 0$. Therefore if $k = 1, 1.5$ then $f$ is continuous at the origin. For the case $k = 2$ consider paths of the form $(t, mt)$ and get

$$\lim_{t\to 0} f(t, mt) \lim_{t\to 0} \frac{m^3}{(1+m^2)^2}$$

which depends on $m$. Therefore $f$ is not continuous (and in particular not differentiable for $k = 2$).

Next, we check the existence of partial derivatives for $k = 1, 1.5$.

$$f_x(0,0) = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{h^4}{h^{2k}h} = \begin{cases} 0 & k = 1 \\ 1 & k = 1.5 \end{cases}$$

Therefore the partial derivatives exist.

But is it differentiable?

Let's obtain a formula for $\varepsilon(x, y)$. We have,

$$f(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + \varepsilon(x, y)\sqrt{(x - 0)^2 + (y - 0)^2}$$

For $k = 1$ we have

$$\varepsilon(x, y) = \frac{xy^3 + y^4}{(x^2 + y^2)^{3/2}}$$

and in polar coordinates

$$\begin{aligned} |\varepsilon(r\cos\theta, r\sin\theta)| &= \frac{r^4(\sin^3\theta\cos\theta + \sin^4\theta)}{r^3} \\ &\leq 2r \equiv g(r) \xrightarrow{r\to 0} 0 \end{aligned} \tag{7}$$

Therefore $f$ is differentiable for $k = 1$.

For $k = 1.5$ we have

$$\begin{aligned} \varepsilon(x, y) &= \frac{f(x, y) - y}{\sqrt{x^2 + y^2}} \\ &= \frac{xy^3 + y^4}{(x^2 + y^2)^2} - \frac{y}{\sqrt{x^2 + y^2}} \end{aligned} \tag{8}$$

Consider paths of the form $(t, mt)$ and notice that

$$\begin{aligned} \lim_{t\to 0} \varepsilon(t, mt) &= \lim_{t\to 0} \frac{t^4(m^3 + m^4)}{t^4(1 + m^2)^2} - \frac{mt}{\sqrt{t^2(1 + m^2)}} \\ &= \frac{m^3 + m^4}{(1 + m^2)^2} \pm \frac{m}{\sqrt{1 + m^2}} \end{aligned} \tag{9}$$

which depends on $m$. Therefore $\varepsilon \not\to 0$ and $f$ is not differentiable for $k = 1.5$.

**Theorem 4** *If $f_x, f_y$ exist and are continuous at $p_0$ then $f$ is differentiable at $p_0$.*

Notice that the opposite is not true! For example the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}}) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$ but $f_x$ is not continuous at $(0,0)$. Check it!

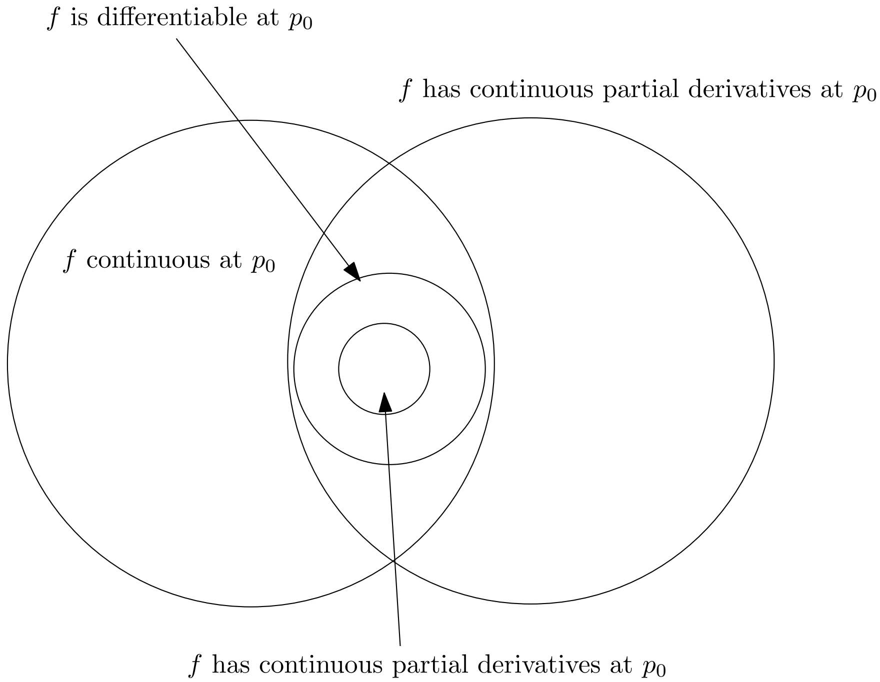


Figure 1: The relationship between the existence of partial derivatives of differentiability

# The chain rule

Recall that for a function of a single-variable,

$$\frac{d}{dx}\left(f(u(x))\right) = \frac{df}{du}\frac{du}{dx}$$

and for example,

$$\frac{d}{dx}\left(sin(\ln(x))\right) = \cos(\ln(x))\frac{1}{x}$$

For functions of two variables we have the following theorem.

**Theorem 5** *If $u, v : \mathbb{R}^2 \to \mathbb{R}$ are differentiable at $p_0 = (x_0, y_0)$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(u_0, v_0 = u(p_0), v(p_0))$ then $f(u(x,y), v(x,y))$ is differentiable at $p_0$ and*

$$\frac{\partial}{\partial x}\left(f(u(x,y), v(x,y))\right) = \frac{\partial f}{\partial u} |_{(u_0,v_0)} \frac{\partial u}{\partial x} |_{(x_0,y_0)} + \frac{\partial f}{\partial v} |_{(u_0,v_0)} \frac{\partial v}{\partial x} |_{(x_0,y_0)} \qquad (10)$$

*and similarly for the derivatives with respect to $y$.*

**Exercise 4** *(from a test) Let $f(x,y)$ be a differentiable function such that*

$$f_x(1,1) = 5 \qquad f_y(1,1) = 0 \qquad f_x(2,3) = 3 \qquad f_y(2,3) = 4$$

*Define*

$$g(x,y) = f(x^2 - y + 2, y^3 - x + 3)$$

*Compute $g_x(1,1)$.*

Denote $u(x,y) = x^2 - y + 2$ and $v(x,y) = y^3 - x + 3$, and here $g(x,y) = f(u(x,y), v(x,y))$. $u, v$ are elementary functions $\Longrightarrow$ differentiable for all $x, y$. Therefore by the chain rule,

$$\frac{\partial g}{\partial x}(x,y) = \frac{\partial f}{\partial u} |_{(u(1,1),v(1,1)} \frac{\partial u}{\partial x} |_{(1,1)} + \frac{\partial f}{\partial v} |_{(u(1,1),v(1,1)} \frac{\partial v}{\partial x} |_{(1,1)}$$

Now notice that $(u(1,1), v(1,1)) = (2,3)$, so we need to differentiate $f$ at this point.

$$\frac{\partial f}{\partial u} |_{(2,3)} = f_x(2,3) = 3$$

$$\frac{\partial f}{\partial u} |_{(2,3)} = f_y(2,3) = 4$$

and

$$\frac{\partial u}{\partial x} |_{(1,1)} = 2x |_{(1,1)} = 2$$

$$\frac{\partial v}{\partial x} |_{(1,1)} = -1 |_{(1,1)} = -1$$

Therefore we finally obtain $g_x(1,1) = 2$.

**Exercise 5** *Consider the surface $z = x^2 + y^2$. Compute its tangent plane at $(0,3)$.*

Define $z = f(x,y) = x^2 + y^2$. $f$ is clearly differentiable, and $\nabla f = (2x, 2y)$. At $(0,3)$ the normal vector to the tangent plane is

$$(-f_x(0,3), -f_y(0,3), 1) = (0, 6, -1)$$

with the point $(0, 3, f(0,3)) = (0, 3, 9)$. Therefore we get

$$6y - z - 9 = 0$$

as the tangent plane.

## Directional derivative

**Definition 6** *Let $\hat{n} = (n_1, n_2)$ be a unit vector. The directional derivative of $f(x,y)$ at $(x_0, y_0)$ in the direction of $\hat{n}$ is*

$$\frac{\partial f}{\partial n}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hn_1, y_0 + hn_2) - f(x_0, y_0)}{h}$$

*assuming the limit exists (and is finite).*

If $\hat{n} = (1,0)$ or $\hat{n} = (0,1)$ we get the partial derivatives $f_x, f_y$ respectively.

**Theorem 7** *If $f$ is differentiable at $(x_0, y_0)$ and $\hat{n}$ is a unit vector, then*

$$\frac{\partial f}{\partial n}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{n}$$

From this theorem

$$\frac{\partial f}{\partial n}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{n} = |\nabla f| \cos\theta$$

and it is clear that the value is maximal for $\theta = 0$. Therefore the gradient points in the direction of maximal ascent!

**Exercise 6** *Compute the directional derivative of*

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

*at $(0,0$ in a general direction.*

By definition we get

$$\lim_{h \to 0} \frac{\frac{h^3 n_1^2 n_2}{h^2(n_1^2+n_2^2)} - 0}{h} = n_1^2 n_2$$

Can we just use the theorem? In this case $\nabla f(0,0) = (0,0)$ and we get $\frac{\partial f}{\partial n} = (0,0) \cdot \hat{n} = 0$. What's the reason for the discrepancy? The theorem requires $f$ to be differentiable at the origin! In this case $\varepsilon(x,y) = \frac{x^2 y}{(x^2+y^2)^{3/2}}$ and you can check with paths of the form $(t, mt)$ that it is not differentiable.

Note that: differentiability $\Longrightarrow$ existence of directional derivatives in all directions! But existence of directional derivatives in all directions $\not\Longrightarrow$ differentiability !

**Exercise 7** *Let $f(x,y) = x^3 y - y^3 x$ and $\vec{n} = (1,1)$. Then $\frac{\partial f}{\partial n}(1,2)$ is:*

1. *does not exist*
2. $-\frac{13}{\sqrt{2}}$
3. *13*
4. *-13*

A typical solution: Since $f$ is a polynomial, $f_x, f_y$ exist and are continuous, therefore $f$ is differentiable.

$$\frac{\partial f}{\partial n}(1,2) = \nabla f(1,2) \cdot \vec{n} = (-2,-11) \cdot (1,1) = -13$$

This is wrong!!! We must first normalize $\vec{n}$ to get $\hat{n} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and then the answer is $-13/\sqrt{2}$.

**Exercise 8** *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be a differentiable function such that*

- *For all* $x, y$ *we have* $f(x, y, 2x^2 + y^2) = 3x - 5y$.
- $\frac{\partial f}{\partial n} = 1$ *for* $\hat{n} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

*Compute* $\nabla f(1, 2, 6)$.

We need the partials $f_x, f_y, f_z$ at the point $(1, 2, 6)$. Notice it is a point of the form $(x, y, 2x^2 + y^2)$ for $x = 1, y = 2$.

# A riddle

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