Calculus 2m1
Session 3: Planes and Lines in $\mathbb{R}^3$

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Planes in $\mathbb{R}^3$

Definition 1 *A plane is defined by its normal vector $\vec{n} = (a, b, c)$ and a point through which it passes $\vec{p_0} = (x_0, y_0, z_0)$. Given a point on the plane $\vec{p} = (x, y, z)$, the vector from it to $\vec{p_0}$ must be perpendicular to $\vec{n}$, and therefore*

$$\vec{n} \cdot (\vec{p} - \vec{p_0}) = 0 \tag{1}$$

This defines the equation of the plane. In coordinates,

$$ax + by + cz + d = 0 \tag{2}$$

where d is

$$d = -\vec{n} \cdot \vec{p_0} = -ax_0 - by_0 - cz_0 \tag{3}$$

Exercise 1 *Find the equation of the plane passing through the points $A = (1, 1, 1)$, $B = (1, 2, 3)$ and $C = (3, 1, -1)$.*

We need a normal and a point. The normal is perpendicular to both $\vec{AB}$ and $\vec{AC}$ and therefore $\vec{n} = \vec{AB} \times \vec{AC} = (-2, 4, -2)$. The equation of the plane is $-2(x - x_0) + 4(y - y_0) - 2(z - z_0) = 0$ where $\vec{p_0} = (x_0, y_0, z_0)$ is some point on the plane. Take $\vec{p_0} = \vec{A}$ and we get

$$-2(x - 1) + 4(y - 1) - 2(z - 1) = 0 \tag{4}$$

or by foiling and dividing by -2:

$$x - y + z = 0 \tag{5}$$

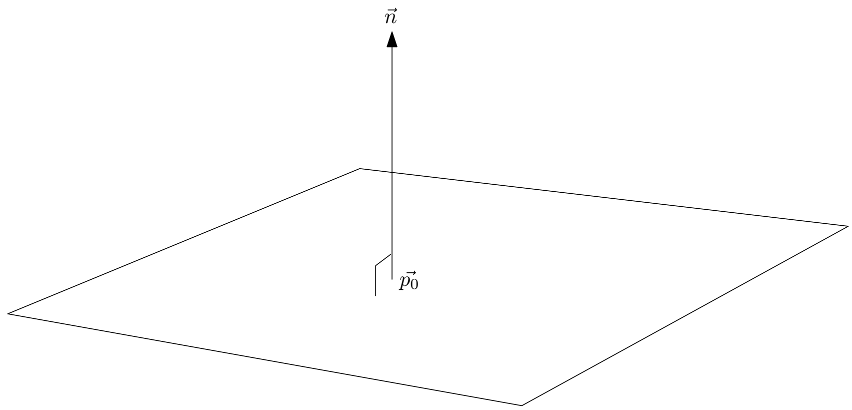

Figure 1: A plane is defined by a point on it p_0 and a normal vector $\vec{n}$.

Definition 2 *The distance between a point $\vec{q} = (x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by the length of the projection of the vector connecting the plane to the point onto the normal to the plane $\vec{n}$. If $\vec{p_0}$ is a point on the plane, then:*

$$p_{\vec{n}}(\vec{q} - \vec{p_0}) = \frac{(\vec{q} - \vec{p_0}) \cdot \vec{n}}{|\vec{c}|} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{6}$$

Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ will intersect (distance $= 0$) $\iff \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Exercise 2 *Find the distance between the planes $2x - 2y + z + 3 = 0$ and $2x - 2y + z - 1 = 0$.*

First notice that their normal vectors are proportional to one another but their are not the same planes $\implies$ the distance is non-zero. To find the distance, take a random point on the first plane and compute its distance to the second plane. For example if the point is of the form $(0, 0, z_0)$ then plugging it in the equation of the first plane gives $z_0 = -3$, namely $\vec{q} = (0, 0, -3)$. Then,

$$\text{distance} = \frac{|2 \cdot 0 + (-2) \cdot 0 + 1 \cdot (-3) - 1|}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{4}{3} \tag{7}$$

Exercise 3 *Find the equation of a plane parallel to $x + 2y - 2z + 1 = 0$ with distance 4 from it.*

There are two answers to this problem. The equation of the parallel plane is of the form $x + 2y - 2z + d = 0$. Consider a point from the first plane, and lets set its distance from our plane to be 4. If the point is of the form $(0,0,z_0)$ then we get $-2z_0+1=0$ so $z_0=1/2$. Its distance from the new plane must satisfy

$$4=\frac{|1\cdot 0+2\cdot 0-2\cdot\frac{1}{2}+d}{\sqrt{1^2+2^2+(-2)^2}}=\frac{|d-1|}{3} \tag{8}$$

therefore $|d-1|=12$ and $d=-11,13$, giving two planes as we expected

$$\begin{aligned} x+2y-2z-11 &= 0 \\ x+2y-2z+13 &= 0 \end{aligned} \tag{9}$$

Lines in $\mathbb{R}^3$

Just like in the plane $\mathbb{R}^2$, a line is defined by a point on it and its slope. Given a line with a slope (l,m,n) and passing through $\vec{p_0}=(x_0,y_0,z_0)$ there are two representations.

Definition 3 *The parametric representation is*

$$\begin{aligned} x(t)=x_0 &\quad+\quad tl \\ y(t)=y_0 &\quad+\quad tm \\ z(t)=z_0 &\quad+\quad tn \end{aligned} \tag{10}$$

namely $(x_0,y_0,z_0)+t(l,m,n)$, *where* (l,m,n) *is called the direction vector.*

Definition 4 *The canonical representation is*

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n} \tag{11}$$

when $l,m,n\neq 0$. *If one of them vanishes, for example if* $l=0$ *we write*

$$x-x_0=0,\frac{y-y_0}{m}=\frac{z-z_0}{n} \tag{12}$$

Exercise 4 *Find the intersection between the planes* $x-2y+z-1=0$ *and* $3x+4y-z=0$.

These planes are not parallel $\Longrightarrow$ indeed have a non-trivial intersection, giving us a line in $\mathbb{R}^3$. To find its equation we need a slope and a point on it. The line is on both planes and therefore is perpendicular to both. If their normal vectors are $\vec{n_1}=(1,-2,1)$ and $\vec{n_2}=(3,4,-1)$ then its direction vector is

$$\vec{n_1}\times\vec{n_2}=2(-1,2,5) \tag{13}$$

so we may take the direction vector to be $(-1, 2, 5)$. To get a point, we set $x = 0$ (for example...) and get two equations for two unknowns:

$$\begin{aligned} -2y + z - 1 &= 0 \\ 4y - z &= 0 \end{aligned} \tag{14}$$

giving $y = 1/2$ and $z = 2$, so the point is $(0, 1/2, 2)$ and the line is given by

$$(0, \frac{1}{2}, 2) + t(-1, 2, 5) \tag{15}$$

or

$$-x = \frac{y - 1/2}{2} = \frac{z - 2}{5} \tag{16}$$

The relationship between two lines

Say we have to lines $(x_1, y_1, z_1) + t(l_1, m_1, n_1)$ and $(x_2, y_2, z_2) + t(l_2, m_2, n_2)$. There are two situations:

1. If they are in the same direction, then there exists α such that $(l_1, m_1, n_1) = \alpha(l_2, m_2, n_2)$. In this case they are either parallel but two different lines (no point in common) or are the same line (have all points in common).

2. If they are not in the same direction, they either intersect (one point in common) or don't intersect - MITZTALVIM (no points in common).

Definition 5 *The distance between a point $\vec{q} = (x_1, y_1, z_1)$ and a line $\vec{p} + t\vec{n}$ is given by*

$$d = |\vec{pq} \times \hat{n}| = \frac{|\vec{pq} \times \vec{n}|}{|\vec{n}|} \tag{17}$$

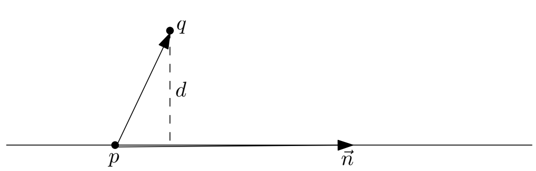

Figure 2: The distance between a point q and a line.

If the lines are parallel, you can just take a point on one of them and use the same formula.

Exercise 5 *(from a test) Given two lines:*

$$x - 2 = \frac{z}{2}, y = 4 \tag{18}$$

and

$$\frac{x-7}{3} = \frac{y-2}{-2} = z - 5 \tag{19}$$

What is the mutual situation between the lines (intersect / parallel / ...)? What is the distance between them?

Writing them parametrically, they are:

$$\begin{aligned} x(t) &= 2 + t \\ y(t) &= 4 \\ z(t) &= 0 + 2t \end{aligned} \tag{20}$$

and

$$\begin{aligned} x(s) &= 7 + 3s \\ y(s) &= 2 - 2s \\ z(s) &= 5 + s \end{aligned} \tag{21}$$

with direction vectors $(1, 0, 2)$ and $(3, -2, 1)$ respectively. It is obvious they are not parallel as they don't have proportional direction vectors.

Do they have a point in common? This will happen if there are t, s such that

$$(2 + t, 4, 2t) = (7 + 3s, 2 - 2s, 5 + s) \tag{22}$$

Check component by component. The second component $\implies 4 = 2-2s$, or $s = -1$. Plugging this into the first component gives $2 + t = 7 - 3 = 4$ so $t = 2$. Plugging this into the third component gives $4 = 5 - 1$ and therefore they do intersect! The point of intersection is $(4, 4, 4)$, and therefore the distance between them is 0.

Definition 6
- $\vec{p_1} = (x_1, y_1, z_1)$ *with direction vector* $\vec{n_1} = l_1, m_1, n_1)$
- $\vec{p_1} = (x_2, y_2, z_2)$ *with direction vector* $\vec{n_1} = l_2, m_2, n_2)$

The distance between them is

$$d = \frac{|p_1\vec{p}_2 \cdot (\vec{n_1} \times \vec{n_2})|}{|\vec{n_1} \times \vec{n_2}|} = \frac{\textit{Volume of parallelogram}}{\textit{Area of parallelogram}} = \textit{height of parallelogram} \tag{23}$$

Definition 7 *Given two non-intersecting lines:*

- $\vec{p_1} = (x_1, y_1, z_1)$ *with direction vector* $\vec{n_1} = l_1, m_1, n_1)$
- $\vec{p_1} = (x_2, y_2, z_2)$ *with direction vector* $\vec{n_1} = l_2, m_2, n_2)$

The distance between them is

$$d = \frac{|\vec{p_1 p_2} \cdot (\vec{n_1} \times \vec{n_2})|}{|\vec{n_1} \times \vec{n_2}|} = \frac{\textit{Volume of parallelogram}}{\textit{Area of parallelogram}} = \textit{height of parallelogram} \quad (24)$$

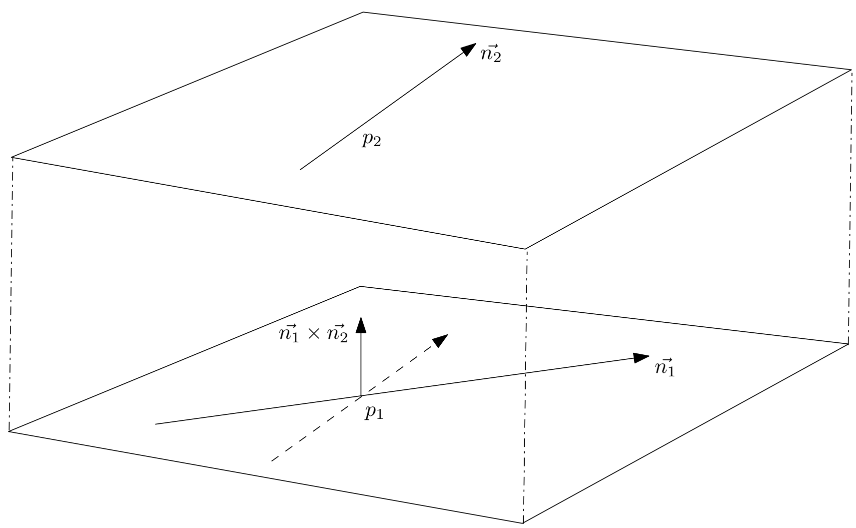

Figure 3: The distance between two lines.

Topology

Let U be a set in $\mathbb{R}^n$.

- An (open) *ball* in $\mathbb{R}^n$ around the point p with radius r is
$$B(p, r) = \{x \in \mathbb{R}^n : |p - x| < r\} \quad (25)$$
- The *boundary* of U (denoted ∂U) is the set of all the points that any open ball around them contains points of U and points of U^c.
- U is an *open set* if for all $p \in U$ there exists $r > 0$ such that $B(p, r) \subseteq U$. In particular, if U is open then it has no boundary points ($\partial U \not\subseteq U$).
- U is a *closed set* if U^c is open. In particular, $\partial U \subseteq U$ implies U is closed.
- U is *bounded* if there exists $r > 0$ such that $U \subseteq B(0, r)$.
- U is *connected* if between any two points in U we can find a continuous curve.

A riddle

Using 6 sticks with identical sizes, create 4 equilateral triangles without having modifying the sticks (or having any extra parts popping out).