Complex functions and integral transformations
Session 1: Introduction

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- know three vectors and the volume of the parallelogram. Now look at three combos of the vectors and compute their products. - projection (e.g. onto the main diagonal of a cube).

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Exercise 1 *1. Find a (the only...) unit vector on the plane in the direction of the positive main diagonal.*

2. Find c for which $\vec{v} = c\hat{i} - \hat{j}$ is perpendicular to it.

The direction is $\hat{i} + \hat{j}$. Normalize it to get $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$. Now its scalar product with $\vec{v}$ is $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j}) \cdot (c\hat{i} - \hat{j}) = \frac{1}{\sqrt{2}}(c-1)$. The vectors are perpendicular for $c = 1$.

Exercise 2 *Find a point on the line segment connecting $A = (1,2)$ and $B = (7,11)$ that is distanced one third the way from A to the point B.*

The line segment connecting a point A and B is defined by the parameterization $\gamma(t) = \vec{A} + (\vec{B} - \vec{A})t$ (so $\gamma(0) = \vec{A}$ and $\gamma(1) = \vec{B}$). In our case, the parameterization is $\gamma(t) = (1,2) + (6,9)t$, so that $\vec{C} = \gamma(1/3) = (1,2) + (2,3) = (3,5)$. Check: $\|\vec{AB}\} = \sqrt{6^2 + 9^2} = \sqrt{117}$ while $\|\vec{CA}\| = \sqrt{2^2 + 3^2} = \sqrt{13}$ and indeed $13/117 = 1/9$ so it works (take sort).

Exercise 3 *(from a quiz, Winter 2008-2009) Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors that satisfy $\|\vec{a}\| = 1$, $\|\vec{b}\| = 2$, $\|\vec{c}\| = 3$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Compute $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.*

We have

$$0 = (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \tag{1}$$

Foiling it gives $0 = 1^2 + 2^2 + 3^2 + 2x$ where $x = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. Therefore $x = -14/2 = -7$.

Definition 1 *The vector product between $\vec{u}$ and $\vec{v}$ is defined to be*

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \tag{2}$$

It is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$ (directed according to the right-hand-rule) with length

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\||\sin(\theta)| \tag{3}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. PICTURE!

Notice that $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ (it is anti-commutative unlike the scalar product which is commutative). Geometrically, the length of the vector $\vec{u} \times \vec{v}$ is the area of the parallelogram created by $\vec{u}$ and $\vec{v}$. Moreover, $\vec{u}$ is parallel to $\vec{v}$ if and only if $\vec{u} \times \vec{v} = 0$.

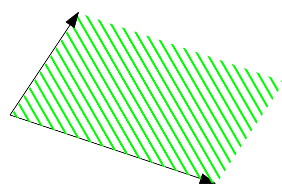


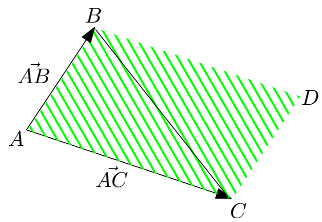
Figure 1: The length of $\vec{u} \times \vec{v}$ is the area of the parallelogram the vectors create.

Exercise 4 *Find the area of a triangle made from vectors $\vec{A}, \vec{B}$ with lengths $\|\vec{A}\| = 6, \|\vec{B}\| = 4$ and scalar product $\vec{A} \cdot \vec{B} = 12$.*

The angle between the vectors satisfies $\cos\theta = \frac{12}{4 \cdot 6} = 1/2$. Therefore $\theta = \pm\pi/3$. This means that the size of their cross product is $\|\vec{A} \times \vec{B}\| = 4 \cdot 6 \sin(\pm\pi/3) = \sqrt{3}12$, and the area of the triangle is $6\sqrt{3}$.

Exercise 5 *Compute the area of a triangle with vertices $A = (1, 2, 3)$, $\vec{B} = (1, 1, -1)$ and $\vec{C} = (2, 4, 3)$.*

The area of the triangle ABC is half the area of the parallelogram $ABDC$. $\vec{AB} = (1-1)\hat{i} + (1-2)\hat{j} + (-1-3)\hat{k}$ and $\vec{AC} = (2-1(\hat{i} + (4-1)\hat{j} + (3+1)\hat{k}$ so there product is $\vec{AB} \times \vec{AC} = 8\hat{i} - 4\hat{j} + 1\hat{k}$. The length of this vector is $\|\vec{AB} \times \vec{AC}\| = \sqrt{81} = 9$ so the area of the triangle is half of this, 4.5.



Definition 2 *The projection of the vector $\vec{u}$ on the vector $\vec{v}$ is*

$$p_{\vec{v}}(\vec{u}) = (\|\vec{u}\| \cos\theta)\hat{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}\hat{v} = (\hat{v} \cdot \vec{u})\hat{v} \tag{4}$$

where $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$ is the direction of $\vec{v}$.

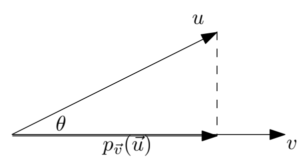


Figure 2: The projection of $\vec{u}$ on $\vec{v}$

Definition 3 *The triple product of $\vec{u}, \vec{v}, \vec{w}$ is*

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \tag{5}$$

The result is a scalar, and notice that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$. Geometrically, its absolute value gives the volume of a parallelogram spanned by the three vectors.

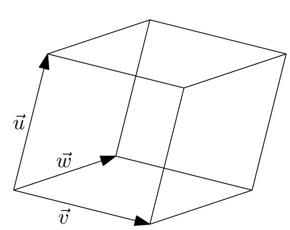


Figure 3: the absolute value of the triple product gives the volume of a $3D$ parallelogram.

The triple product $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ if and only if $\vec{u}, \vec{v}, \vec{w}$ are on the same plane (in which case they are called collinear).

Exercise 6 *Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors in $\mathbb{R}^2$. Let $\vec{v} \neq \vec{0}$ be a vector such that $\vec{v} \cdot \vec{a} = 1$, $\vec{v} \cdot \vec{b} = \vec{v} \cdot \vec{c} = 0$. Proof that if $\lambda = (\vec{a} \times \vec{b}) \cdot \vec{c}$ then*

$$\lambda \vec{v} = \vec{b} \times \vec{c} \tag{6}$$

Since $\vec{v} \cdot \vec{b} = \vec{v} \cdot \vec{c} = 0$ the vectors $\vec{b}$ and $\vec{c}$ are perpendicular to $\vec{v}$, meaning that (since we are in $\mathbb{R}^3$) $\vec{v}$ is parallel to $\vec{b} \times \vec{c}$, namely there exists $\alpha \in \mathbb{R}$ such that $\alpha \vec{v} = \vec{b} \times \vec{c}$. This means λ is

$$\lambda = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\alpha \vec{v}) = \alpha \tag{7}$$

proving what we wanted.

A riddle

Mr. and Mrs. Cohen invited friends over for dinners. They invited 4 couples. When they got in, they shook hands (no one shook hands with his partner or with himself or herself). During dinner, Mrs. Cohen asked how many people they shook hands with. She got 9 different answers. With how many people did Mr. Cohen shake hands?