Complex functions and integral transformations
Session 1: Introduction

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October 12, 2013

Introduction, office hour, course website, my website. This is by far the most useful course in mathematics to applications!

Lines

Reminder: a linear equation is an equation of the form $y = ax + b$ where

- b is the intersection of the line with the y-axis.
- a is the slope, namely the ratio $a = \frac{\Delta y}{\Delta x}$.

If $a > 0$ the line is increasing, and if $a < 0$ it is decreasing, where $a = \tan\theta$ (see figure 1).

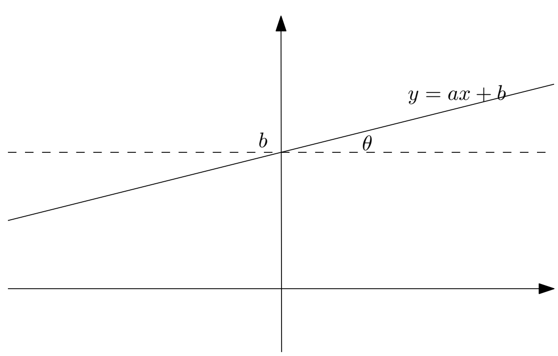
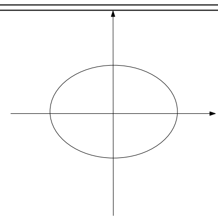
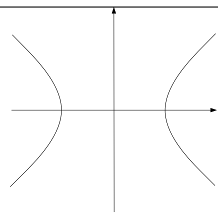
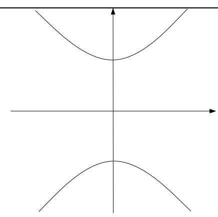
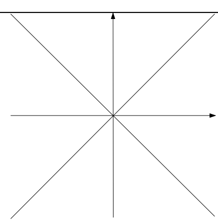
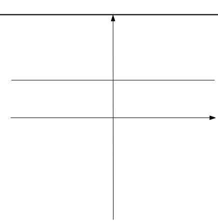


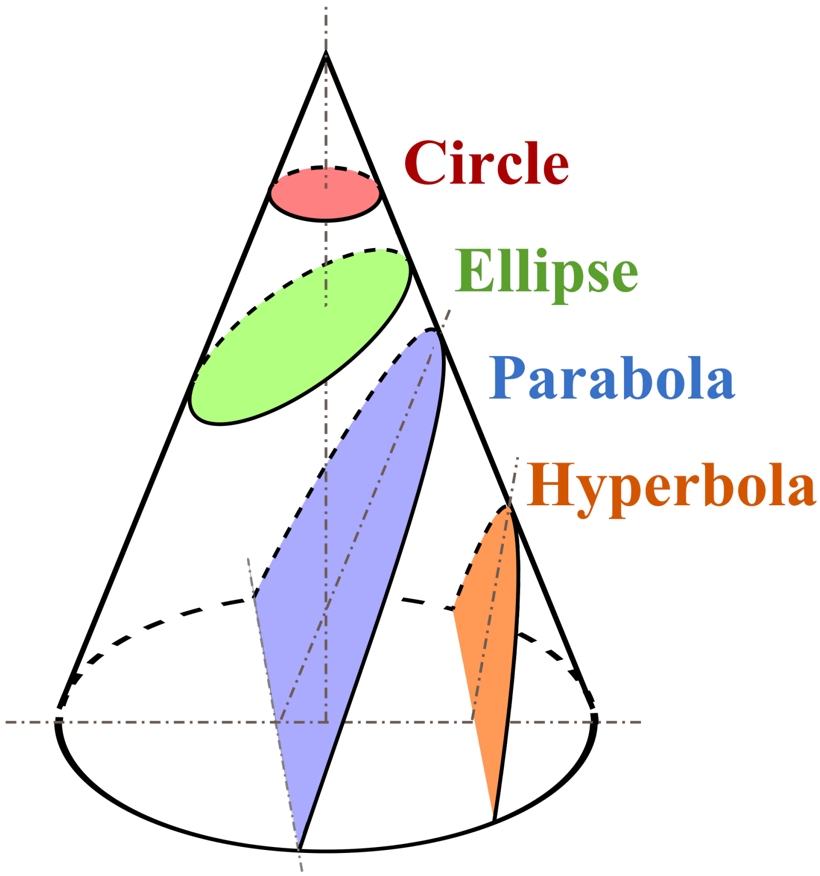
Figure 1: A line

Quadratic forms (curves)

Consider a quadratic form $Ax^2 + By^2 = C$. It may represent different objects, summarized in table X.

A	B	C	$Figure$	$Canonical form$	$Remarks$
$+$ $-$	$+$ $-$	$+$ $-$		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$A = B$ or $a = b$ gives a circle
$+$ $-$	$-$ $+$	$+$ $-$		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	undefined for $x = 0$
$-$ $+$	$+$ $-$	$+$ $-$		$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	undefined for $y = 0$
$+$ $-$ 0 0	$+$ $-$ $+$ $-$	$-$ $+$ $-$ $+$	$\emptyset$	None	undefined!
$+$ $-$	$-$ $+$	0 0			$y = \pm\sqrt{\frac{A}{B}}x$
0 0	$+$ $-$	$+$ $-$			$y = \pm\sqrt{\frac{C}{B}}$
0 0	$+$ $-$	0 0	x-axis	$y = 0$	
$+$ $-$	$+$ $-$	0 0	the origin $(0, 0)$		

They are all special types of conic sections! Very useful in applications. In astronomy, they describe the motion of the two body problem (for two bodies with barycenter, mutual center of mass, which is at rest like a star and a asteroid).



Exercise 1 *Find the shape of the curve* $\alpha(x^2+1)+y^2=1$ *for all values of* α.

Write the equation in the generic form $\alpha x^2 + y^2 = 1-\alpha$. So in the notations above $a=\alpha, b=1, c=1-\alpha$. The coefficients change sign at $\alpha=0,1$. There are five cases:

- $\alpha < 0$ means $a < 0$ and $b, c > 0$ so we get a hyperbola.
- $\alpha = 0$ means $a = 0$ with $b, c > 0$ so we get two horizontal lines $y = \pm 1$.
- $0 < \alpha < 1$ means $a, b, c > 0$ giving an ellipse. It is never a circle $(a \neq b)$!
- $\alpha = 1$ means $a, b > 0$ and $c = 0$ so we get a point.
- $\alpha > 1$ means $a, b > 0$ and $c < 0$ giving an empty set.

Notice the form of the change in the shape is continuous. Changing from a hyperbola at $\alpha < 0$ to two horizontal lines at $\alpha = 0$, to an ellipse when $0 < alpha < 1$ to a single point and then an empty set.

Definition 1 *A parametric curve in $\mathbb{R}^n$ is a function $\gamma : \mathbb{R} \to \mathbb{R}^n$. In the case of curves in 2D this is a function $\gamma(t) = (x(t), y(t))$.*

For example, an ellipse in the canonical form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as a parameterization $\gamma(t) = (a \cos t, b \sin t)$.

Exercise 2 *Find a parametric representation for the curve $(x^2+y^2)^2 = x^2-y^2$ (called Bernoulli's Lemniscate) restricted to $x > 0$. (a historical remark: many people tried to study this object, especially its arc length as it leads to elliptic functions)*

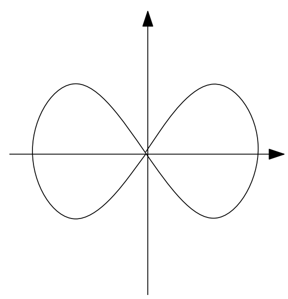


Figure 2: Bernoulli's lemniscate

Switch to polar representation gives $r^4 = r^2(\cos^2\theta - \sin^2\theta)$ so $r = \sqrt{\cos^2\theta - \sin^2\theta} = \sqrt{\cos(2t)}$. It is defined if $\cos(2t) \geq 0$ namely when $-\pi/4 \leq t \leq \pi/4$. The condition $x > 0$ gives $\cos t \geq 0$, namely, $t \in [-\pi/2, \pi/2]$ which is automatically satisfied.

This means that

$$\gamma(t) = (\sqrt{\cos(2t)} \cos t, \sqrt{\cos(2t)} \sin t) \tag{1}$$

with $-\pi/4 \leq t \leq \pi/4$.

Definition 2 *A vector in $\mathbb{R}^n$ is an object with direction and magnitude alone. It is represented by a point $\vec{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where its length is $\|\vec{v}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.*

Note that a vector has no position.

There are several operations on vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$:

- Addition: $\vec{u} + \vec{v} = (u_1 + v_1, \ldots, u_n + v_n)$.

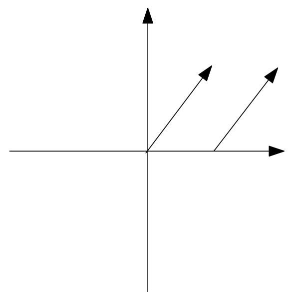
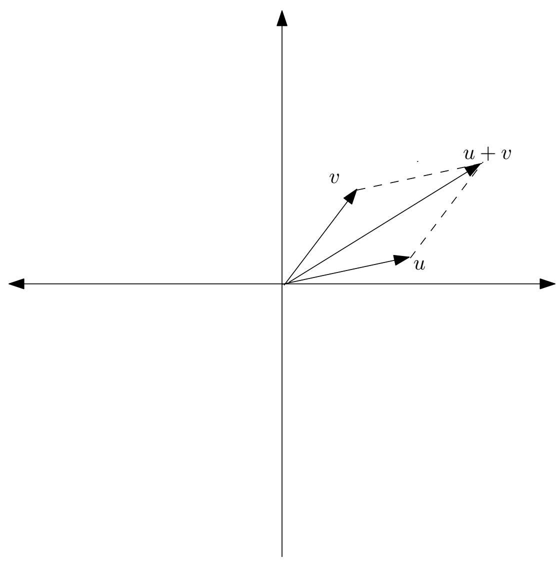


Figure 3: Two equivalent vectors.



- Scalar multiplication: $a\vec{u} = (au_1, \ldots, au_n)$ - rescales the vector (and flips it if $a < 0$).
- Scalar product: $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \ldots u_nv_n = \|u\|\|v\| \cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$ (gives a number, namely a *scalar*!)

If $A, B \in \mathbb{R}^n$ are points, it is common to denote by $\vec{AB}$ the vector that connects them, namely $\vec{AB} = \vec{B} - \vec{A}$.

Notice that two vectors $\vec{u}, \vec{v}$ are perpendicular $\iff \theta = \pm\pi/2 \iff \cos\theta = 0 \iff \vec{u} \cdot \vec{v} = 0$.

Also, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Exercise 3 *Let ABC be the triangle with vertices $A = (0,0)$, $B = (1,1)$ and $C = (1,2)$. Find the length of the median from B to the side AC.*

Let the middle point between A and C be T. Then

$$\vec{BT} = \vec{BC} + \vec{CT} = \vec{BC} + \frac{1}{2}\vec{CA} = \vec{C} - \vec{B} + \frac{1}{2}(\vec{A} - \vec{C}) = (-\frac{1}{2}, 0) \tag{2}$$

so $\|\vec{BT}\| = 1/2$.

The projection of the vector $\vec{u}$ on the vector $\vec{v}$ is

$$p_{\vec{v}}(\vec{u}) = (\|\vec{u}\| \cos\theta)\hat{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}\hat{v} = (\hat{v} \cdot \vec{u})\hat{v} \tag{3}$$

where $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$ is the direction of $\vec{v}$.

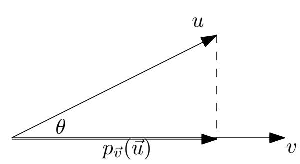


Figure 4: The projection of $\vec{u}$ on $\vec{v}$

Exercise 4 *(from a quiz, Winter 2008-2009) Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors that satisfy $\|\vec{a}\| = 1$, $\|\vec{b}\| = 2$, $\|\vec{c}\| = 3$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Compute $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.*

We have

$$0 = (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \tag{4}$$

Foiling it gives $0 = 1^2 + 2^2 + 3^2 + 2x$ where $x = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. Therefore $x = -14/2 = -7$.

Exercise 5 *Find a point on the line segment connecting $A = (1, 2)$ and $B = (7, 11)$ that is distanced one third the way from A to the point B.*

The line segment connecting a point A and B is defined by the parameterization $\gamma(t) = \vec{A} + (\vec{B} - \vec{A})t$ (so $\gamma(0) = \vec{A}$ and $\gamma(1) = \vec{B}$). In our case, the parameterization is $\gamma(t) = (1, 2) + (6, 9)t$, so that $\vec{C} = \gamma(1/3) = (1, 2) + (2, 3) = (3, 5)$. Check: $\|\vec{AB}\} = \sqrt{6^2 + 9^2} = \sqrt{117}$ while $\|\vec{CA}\| = \sqrt{2^2 + 3^2} = \sqrt{13}$ and indeed $13/117 = 1/9$ so it works (take sort).

A riddle

Take the L-shaped region 1 in figure 5 and split it into two identical parts. Now split region 2 into three identical parts. Last but not least, split region 3 into four identical parts.

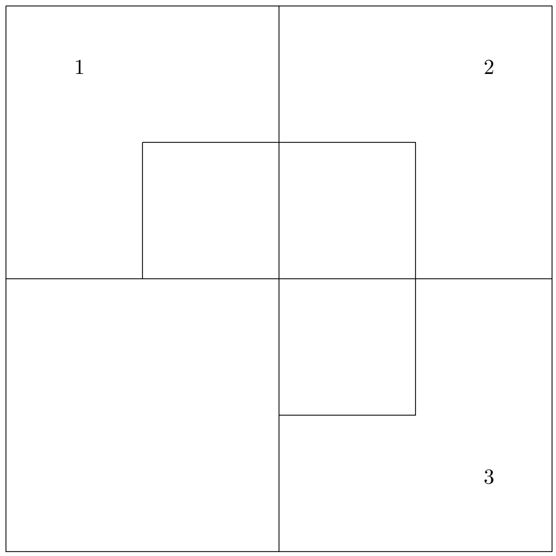


Figure 5: The complex conjugate