General Relativity

Numerical solutions of Einstein's field equation

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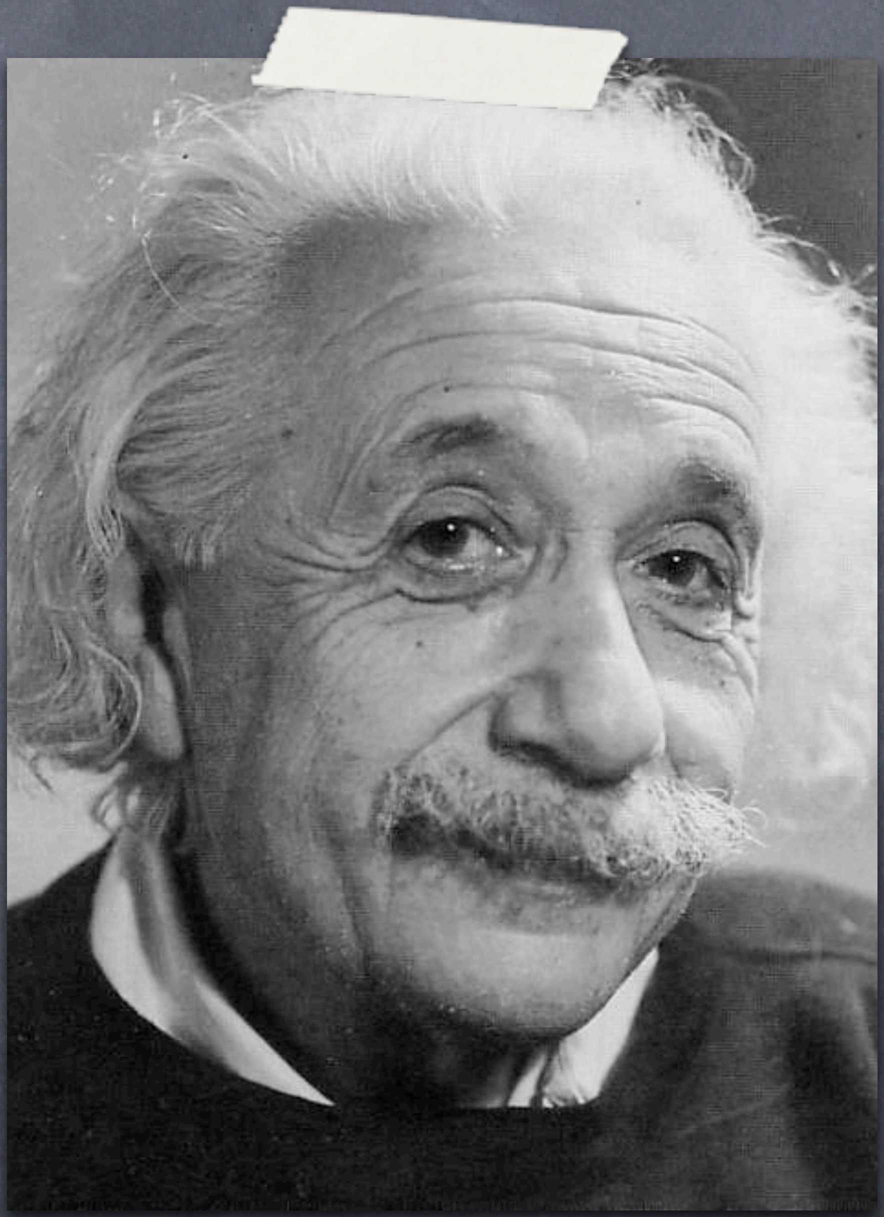
Outline

- Special relativity
- General relativity: gravity as curvature
- Belinskii and Zakharov's solution
- Simulations

Special Relativity (1905)

- Historical background: Maxwell's equations were in conflict with the Galilean transformations.
- Lorentz, Voigt, Larmor and Poincaré found the 'correct' transformations under which Maxwell's equations are invariant.
- Physical interpretation? Here Einstein arrives on the scene.



The constancy of the speed of light

Postulate:

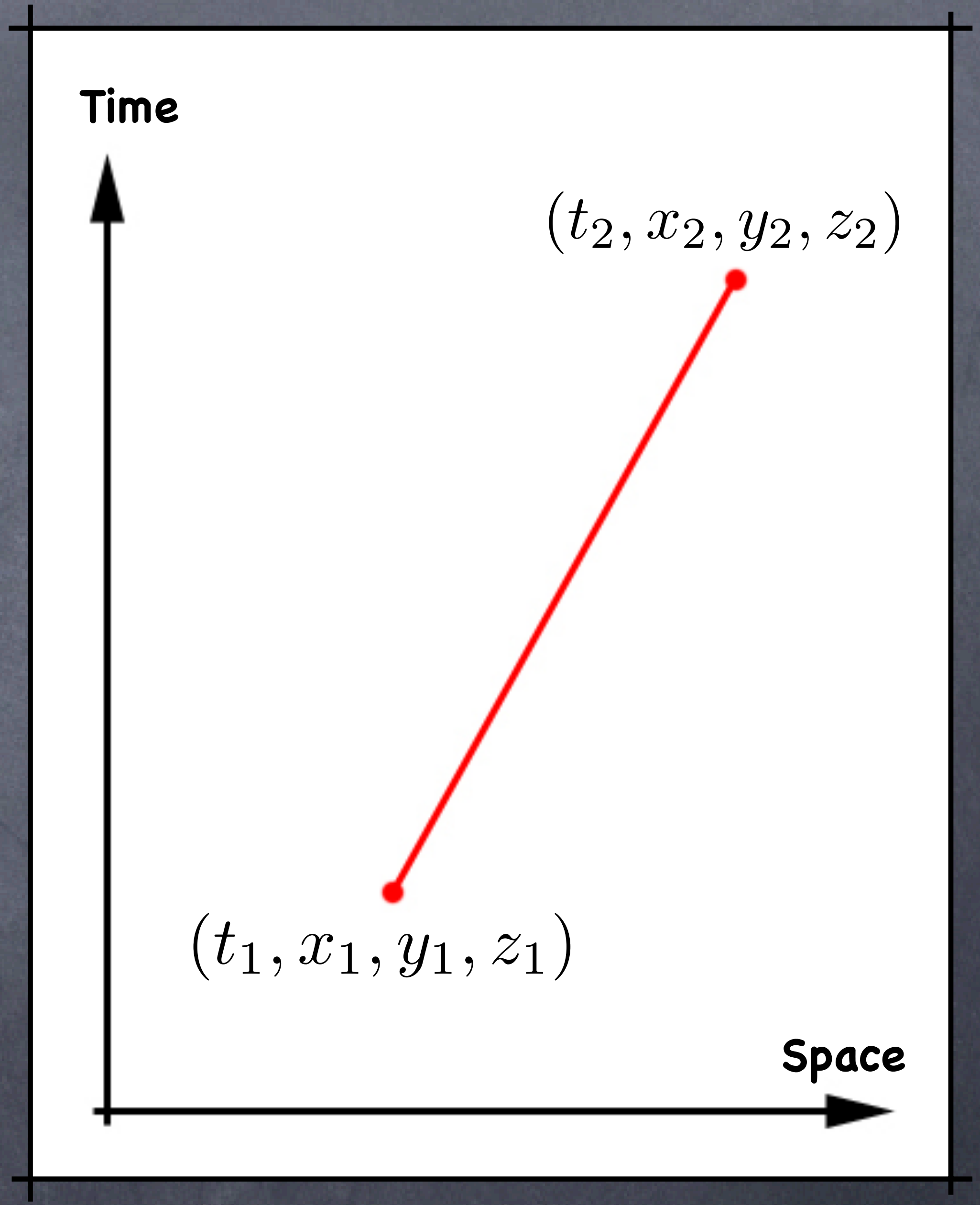
The speed of light c is constant

So for any observer:

$$c\,(dt) = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Or equivalently

$$-c^2\,(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = 0$$

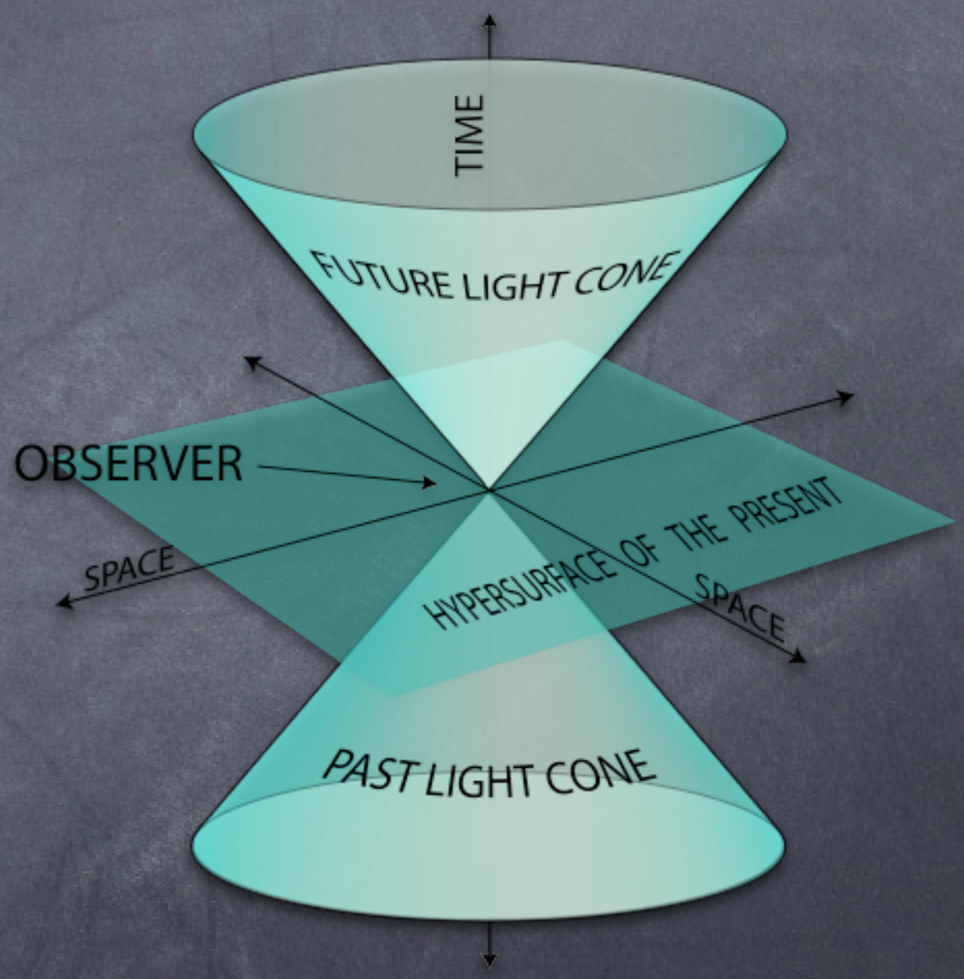


The interval

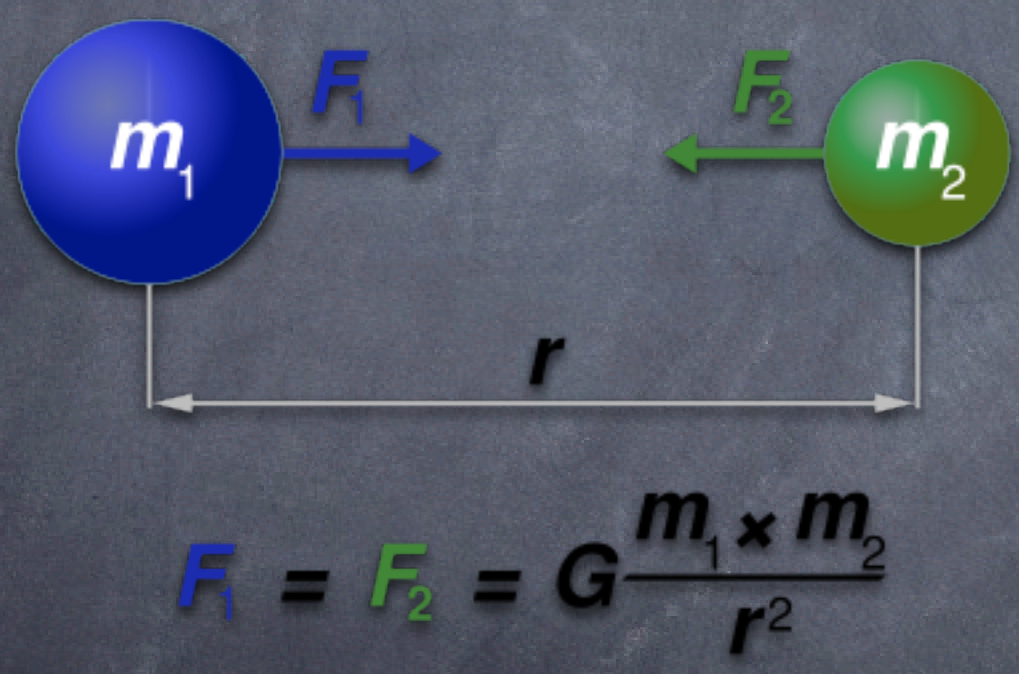
$$ds^2 = -c^2\,(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

$ds^2 > 0$ 'Space-like' events: No causal relation

$ds^2 < 0$ 'Time-like' events: Might have causal relation



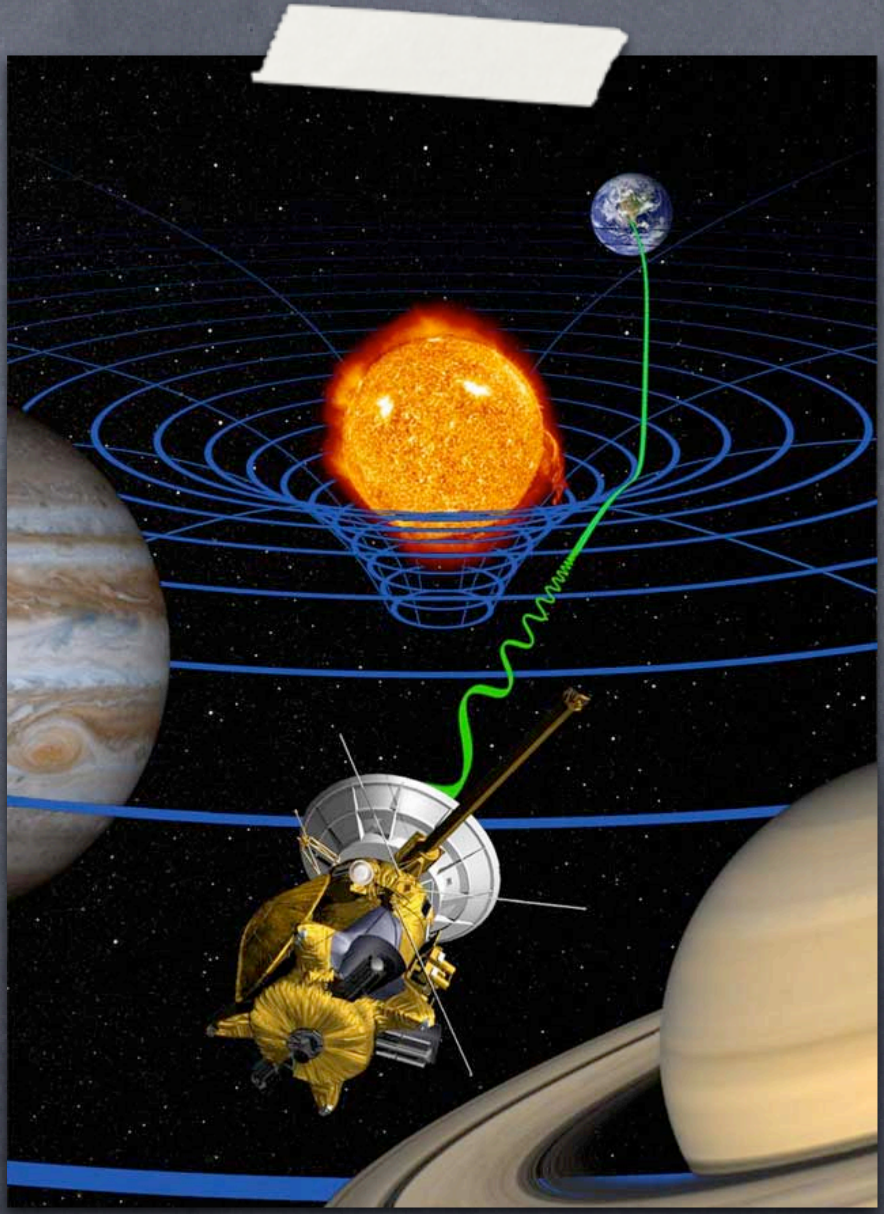
Newton's law of universal gravitation



- Inconsistent with special relativity since it invokes instantaneous influence
- Gravity is an 'action at a distance'

General relativity

- Gravity is not a force anymore!
- Instead, spacetime curves in the presence of matter
- Bodies & light rays travel along geodesics



The interval in general relativity

- The interval in special relativity:

$$ds^2 = -c^2\,(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

- The interval in general relativity:

$$ds^2 = g_{ab}dx^a dx^b$$

where we denote: $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and we sum over repeated indices (a,b=0,1,2,3).

Mathematical formulation of G.R.

- Spacetime is a (Lorentzian) manifold with metric g_{ab}
- Spacetime curves in the presence of matter according to Einstein's field equation:

$$G_{ab} = 8\pi \frac{G}{c^4} T_{ab}$$

"Curvature" = "matter"

The duality between matter and spacetime

So we get a beautiful duality between matter and spacetime:

matter 'tells' spacetime how to curve (Einstein's equation)

Matter

Spacetime

spacetime 'tells' matter how to move (geodesic equation)

Is it that simple?

Nope.

Einstein's equation is extremely nonlinear...
Einstein's equation in all of its glory:

$$
\begin{aligned}
&\partial_c \left[\frac{1}{2} g^{c\beta} \left(\partial_i g_{j\beta} + \partial_j g_{i\beta} - \partial_\beta g_{ij}\right)\right] - \partial_a \left[\frac{1}{2} g^{c\beta} \left(\partial_c g_{j\beta} + \partial_j g_{c\beta} - \partial_\beta g_{cj}\right)\right] + \\
&\left[\frac{1}{2} g^{\alpha\beta} \left(\partial_i g_{j\beta} + \partial_j g_{i\beta} - \partial_\beta g_{ij}\right)\right] \left[\frac{1}{2} g^{c\beta} \left(\partial_\alpha g_{c\beta} + \partial_c g_{\alpha\beta} - \partial_\beta g_{\alpha c}\right)\right] - \\
&\left[\frac{1}{2} g^{\alpha\beta} \left(\partial_c g_{j\beta} + \partial_j g_{c\beta} - \partial_\beta g_{cj}\right)\right] \left[\frac{1}{2} g^{c\beta} \left(\partial_\alpha g_{i\beta} + \partial_i g_{\alpha\beta} - \partial_\beta g_{\alpha i}\right)\right] - \\
&\frac{1}{2} g_{ij} g^{ad} \partial_c \left[\frac{1}{2} g^{c\beta} \left(\partial_a g_{d\beta} + \partial_d g_{a\beta} - \partial_\beta g_{ad}\right)\right] + \frac{1}{2} g_{ij} g^{ad} \partial_a \left[\frac{1}{2} g^{c\beta} \left(\partial_c g_{d\beta} + \partial_d g_{c\beta} - \partial_\beta g_{cd}\right)\right] - \\
&\frac{1}{2} g_{ij} g^{ad} \left[\frac{1}{2} g^{\alpha\beta} \left(\partial_a g_{d\beta} + \partial_d g_{a\beta} - \partial_\beta g_{ad}\right)\right] \left[\frac{1}{2} g^{c\beta} \left(\partial_\alpha g_{c\beta} + \partial_c g_{\alpha\beta} - \partial_\beta g_{\alpha c}\right)\right] + \\
&\frac{1}{2} g_{ij} g^{ad} \left[\frac{1}{2} g^{\alpha\beta} \left(\partial_c g_{d\beta} + \partial_d g_{c\beta} - \partial_\beta g_{cd}\right)\right] \left[\frac{1}{2} g^{c\beta} \left(\partial_\alpha g_{a\beta} + \partial_a g_{\alpha\beta} - \partial_\beta g_{\alpha a}\right)\right] = 8\pi \frac{G}{c^4} T_{ab}
\end{aligned}
$$

The Belinskii and Khalatnikov metric (1969)

- Assume that the metric depends on z and t only.
- Such a metric has many applications, e.g. black holes and cosmological models for the universe.
- It generalizes many other known solutions, e.g. the Schwarzschild and the Kerr solutions.

The metric (Cont'd)

- By a proper coordinate transformation, it can always be written as (a,b=1,2):

$$ds^2 = -f(t,z)(cdt)^2 + g_{ab}(t,z)\,dx^a dx^b + f(t,z)(dz)^2$$

Or in matrix form:

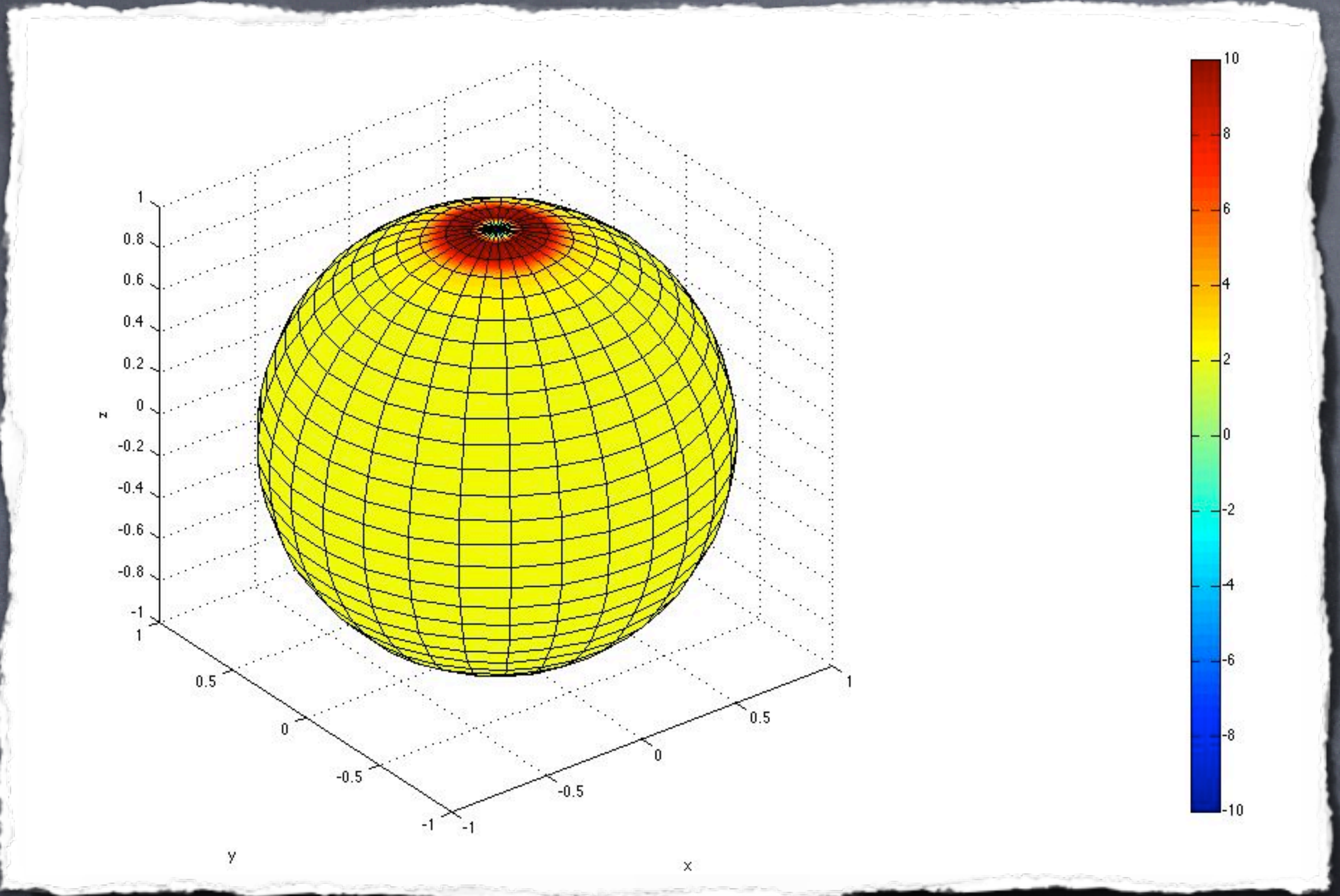
$$g_{ab} = \begin{bmatrix} -f & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$$

Belinskii and Zakharov's solution (1973)

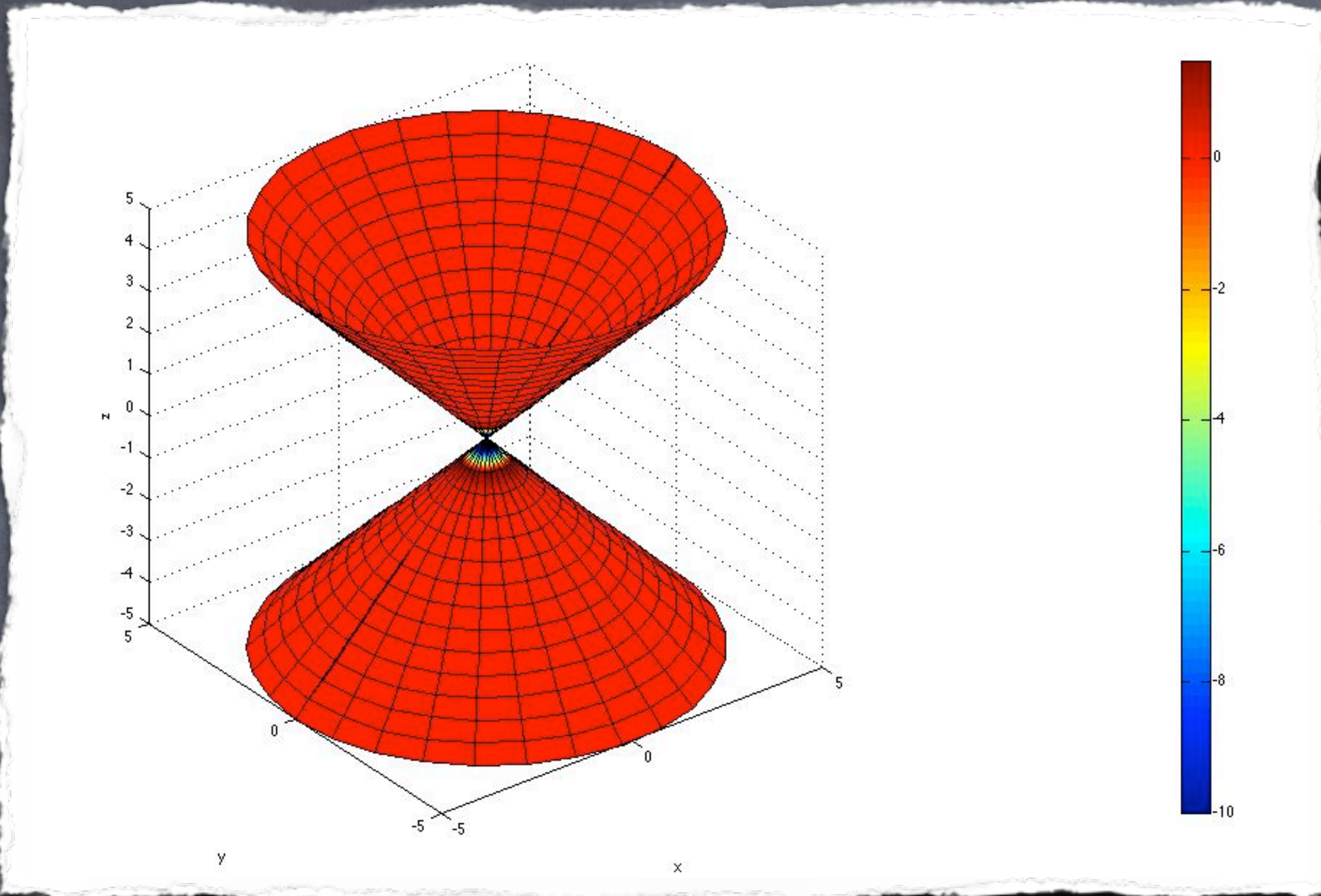
- They applied the inverse scattering method to Einstein's equation.
- Obtained a principal-approximation solution for the Belinskii and Khalatnikov metric.
- The solution is of the form of a gravisoliton: Gravitational wave that travels at the speed of light and maintains its shape.

The project

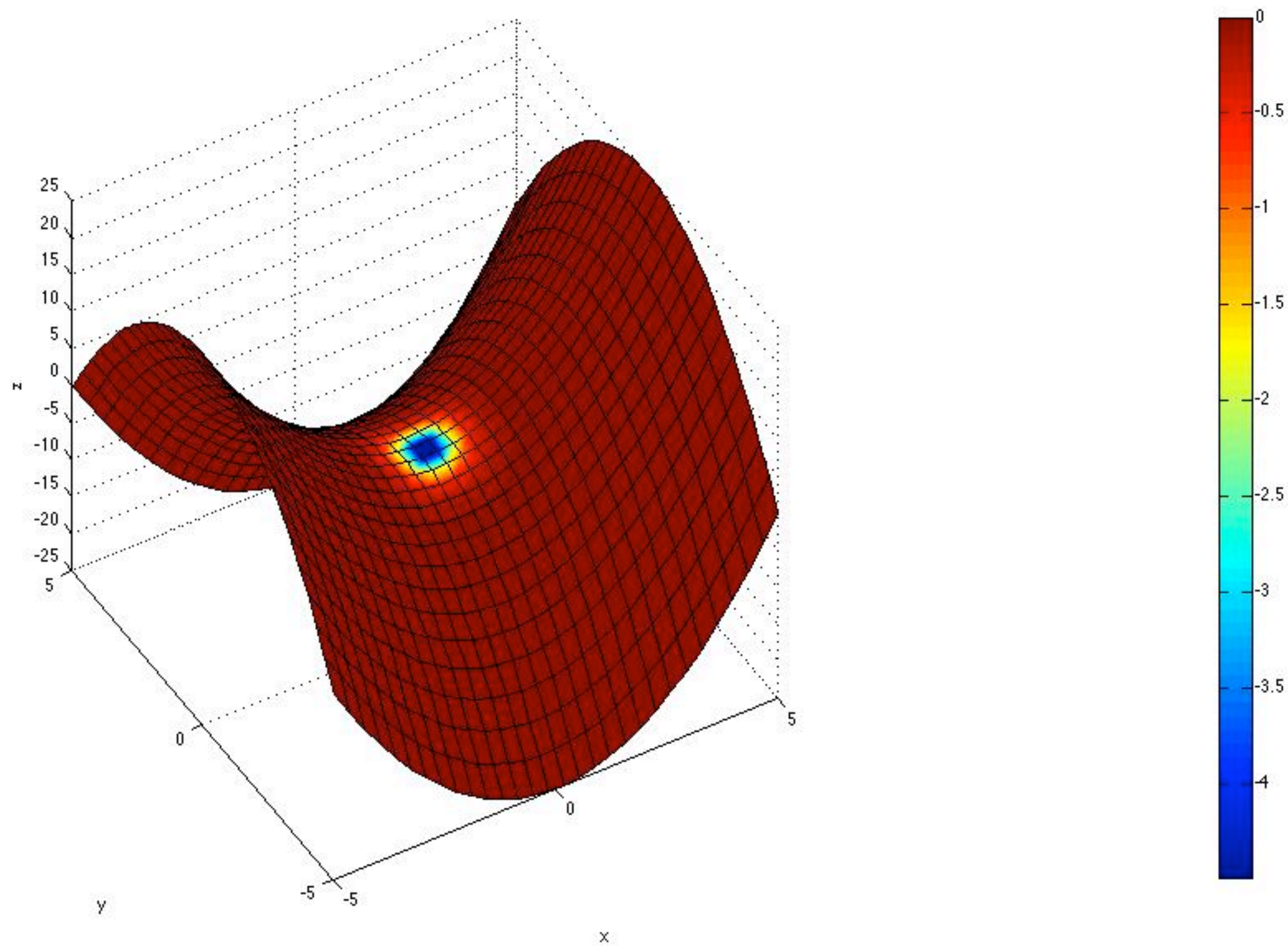
- The goal: Solving Einstein's field equation numerically
- General relativity is based on Riemannian geometry, first simulate curvature!
- The simulations of curvature were verified by checking well-known surfaces (plane, sphere, hyperboloid and etc...)



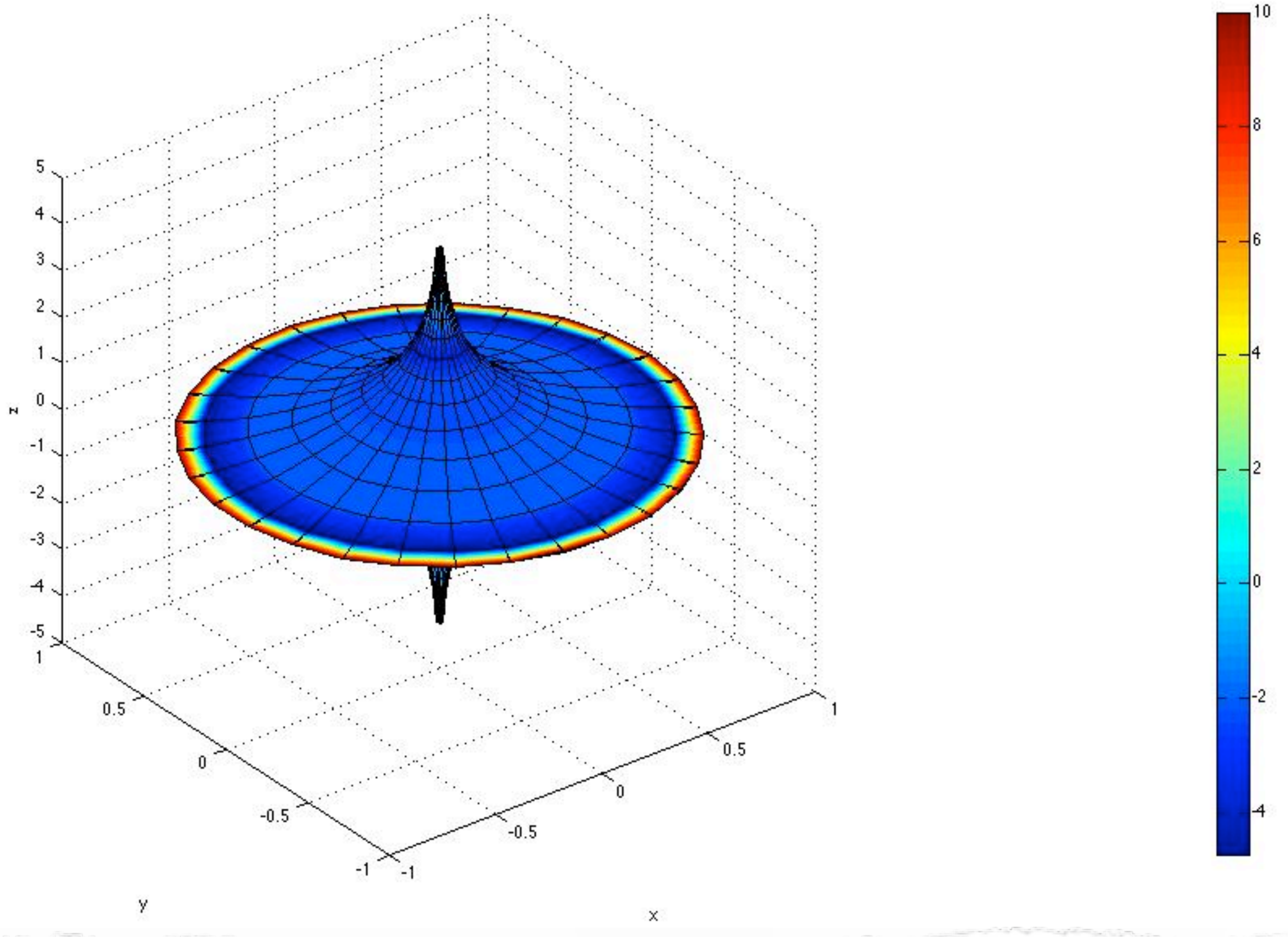
The unit sphere



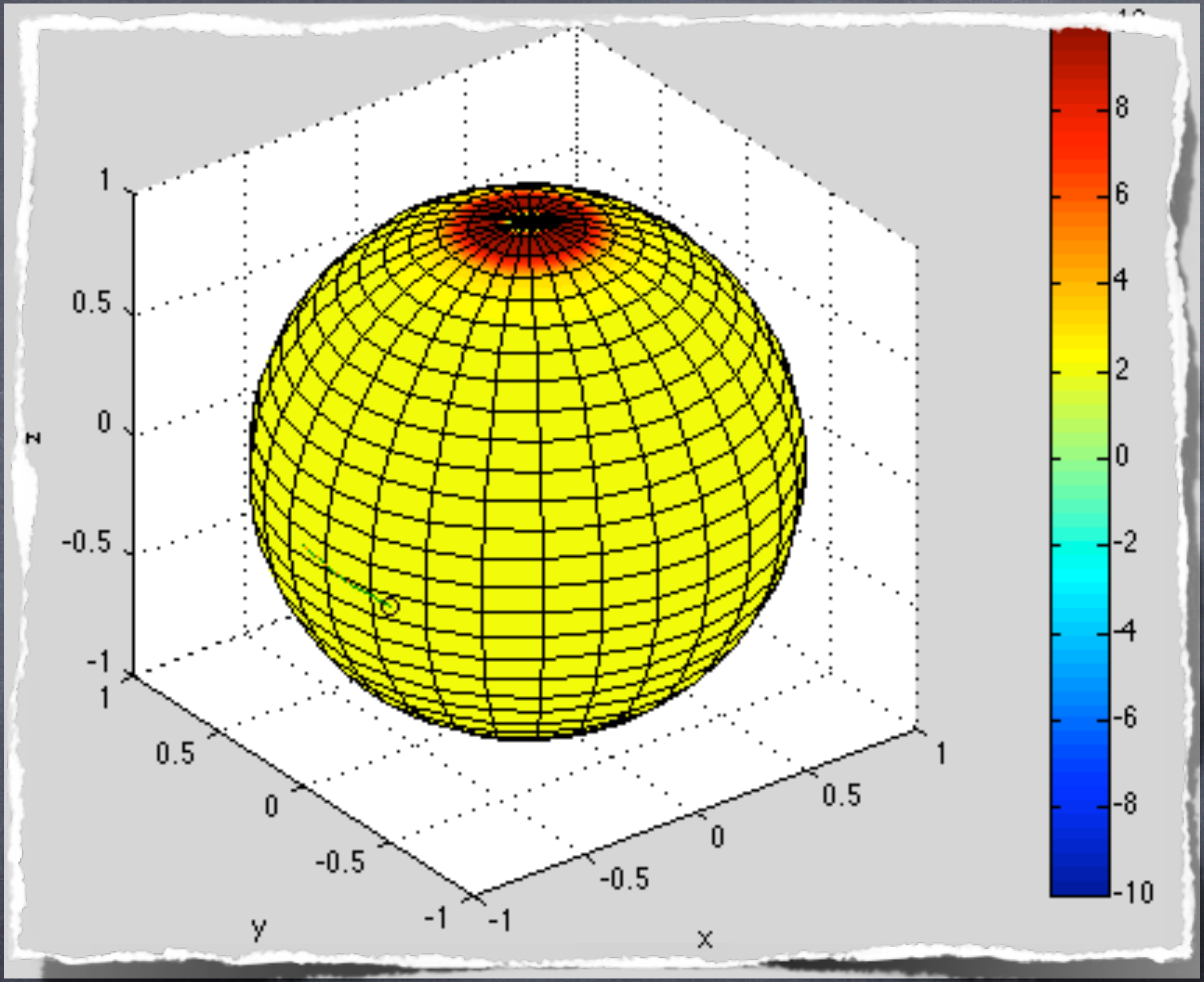
Cone



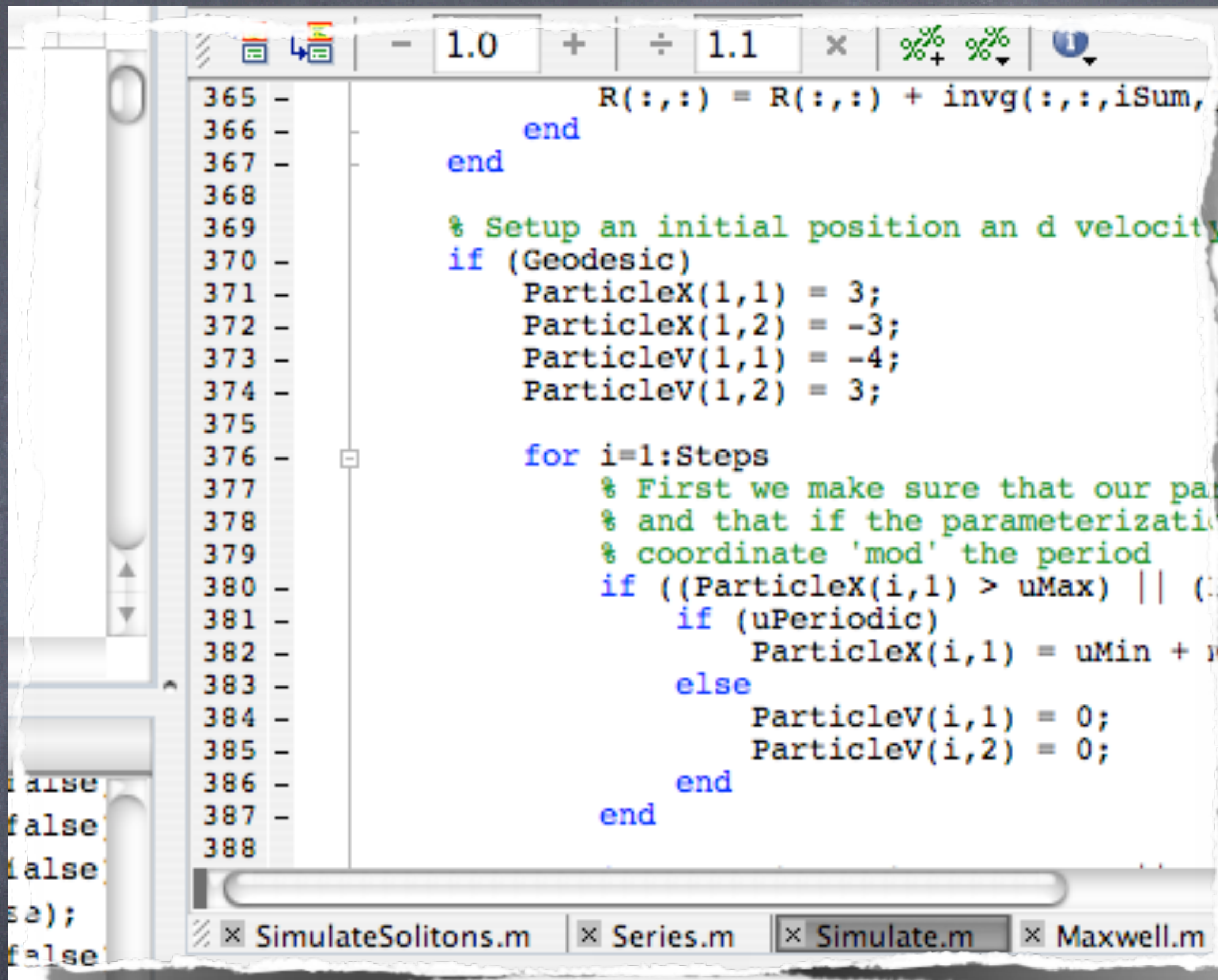
Saddle



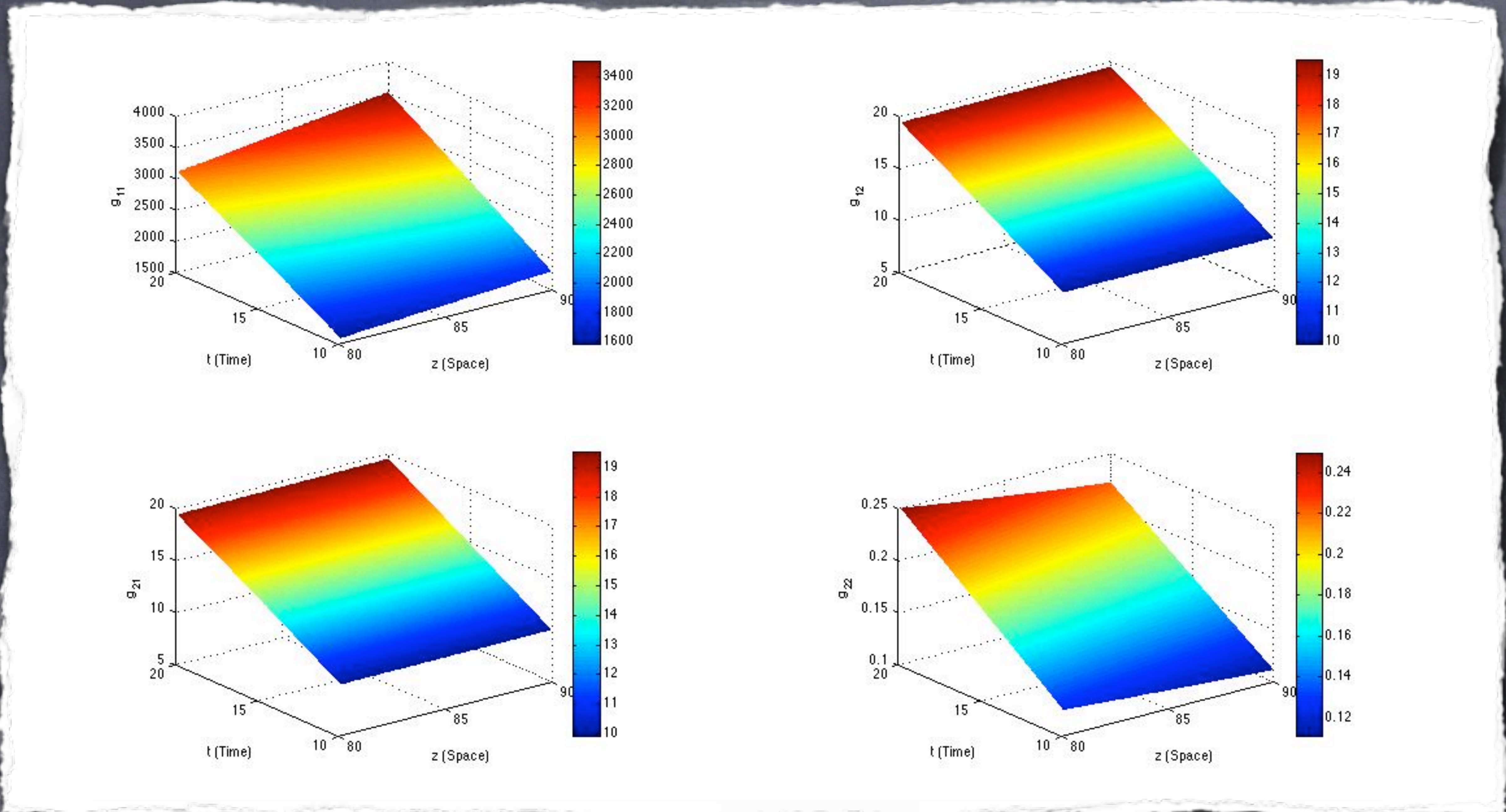
Pseudosphere



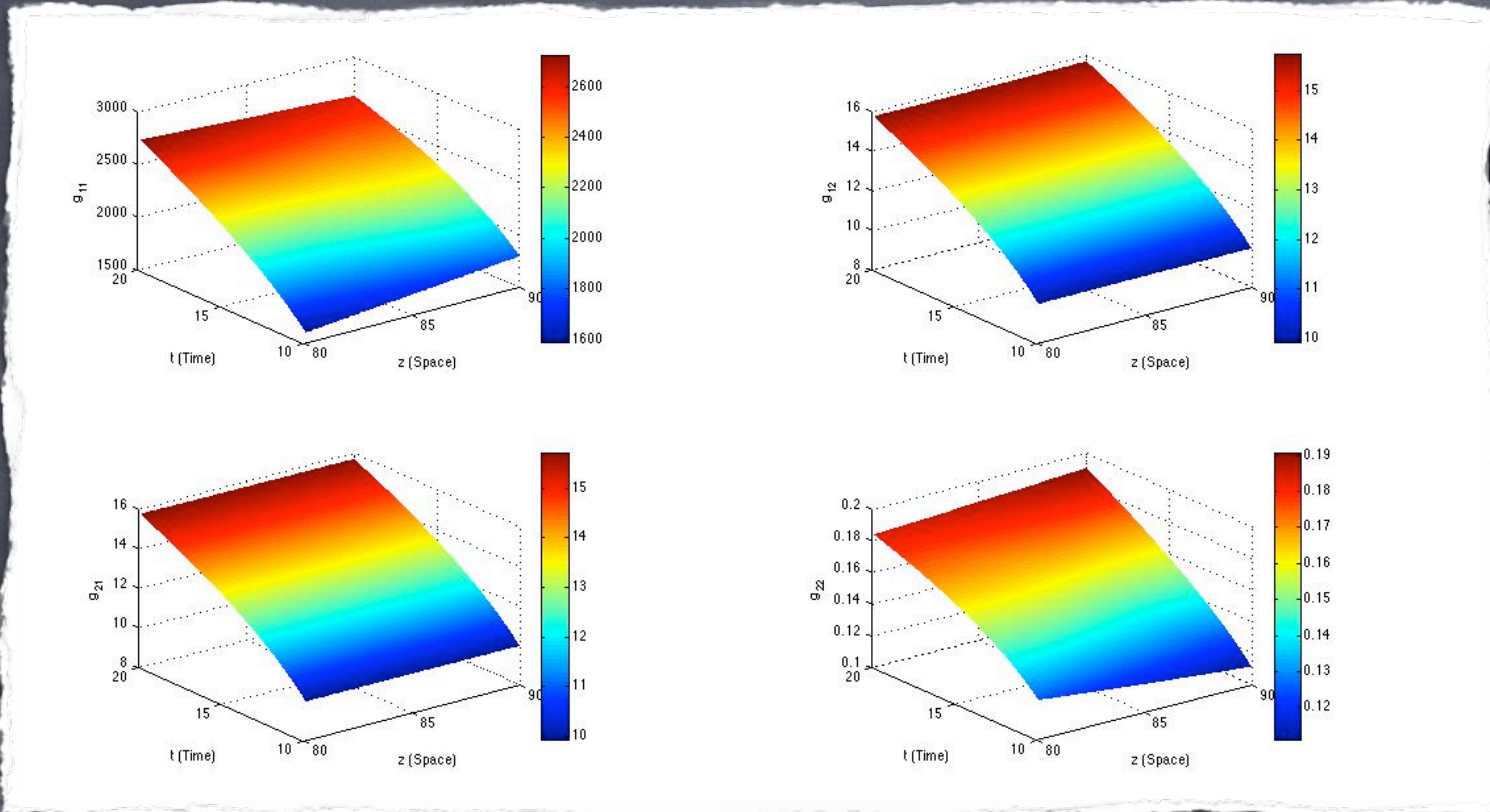
Geodesics: Sphere



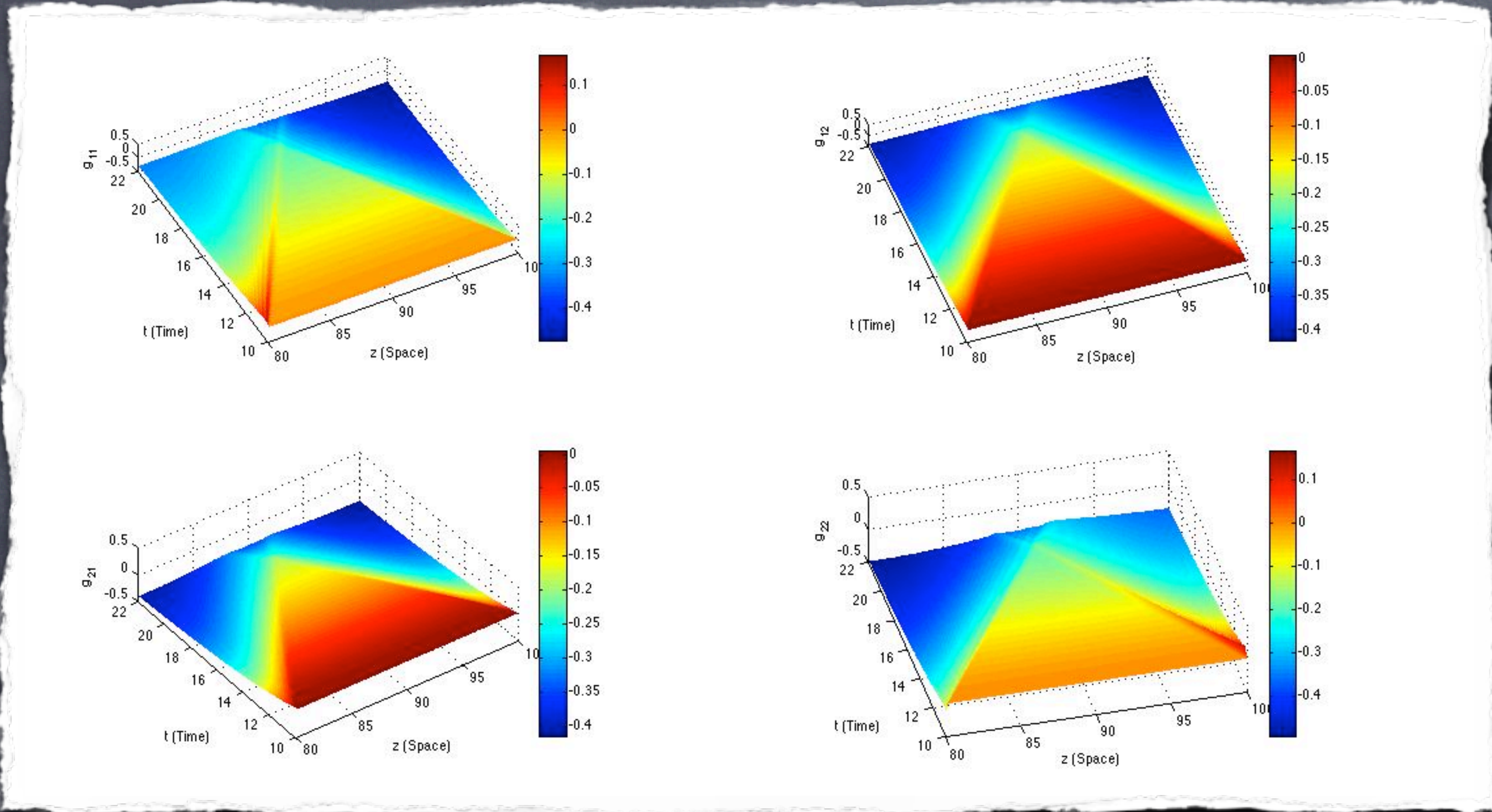
Geodesics: Saddle



The exact solution



The simulated solution



g11
t (Time)
z (Space)
g12
t (Time)
z (Space)
g21
t (Time)
z (Space)
g22
t (Time)
z (Space)

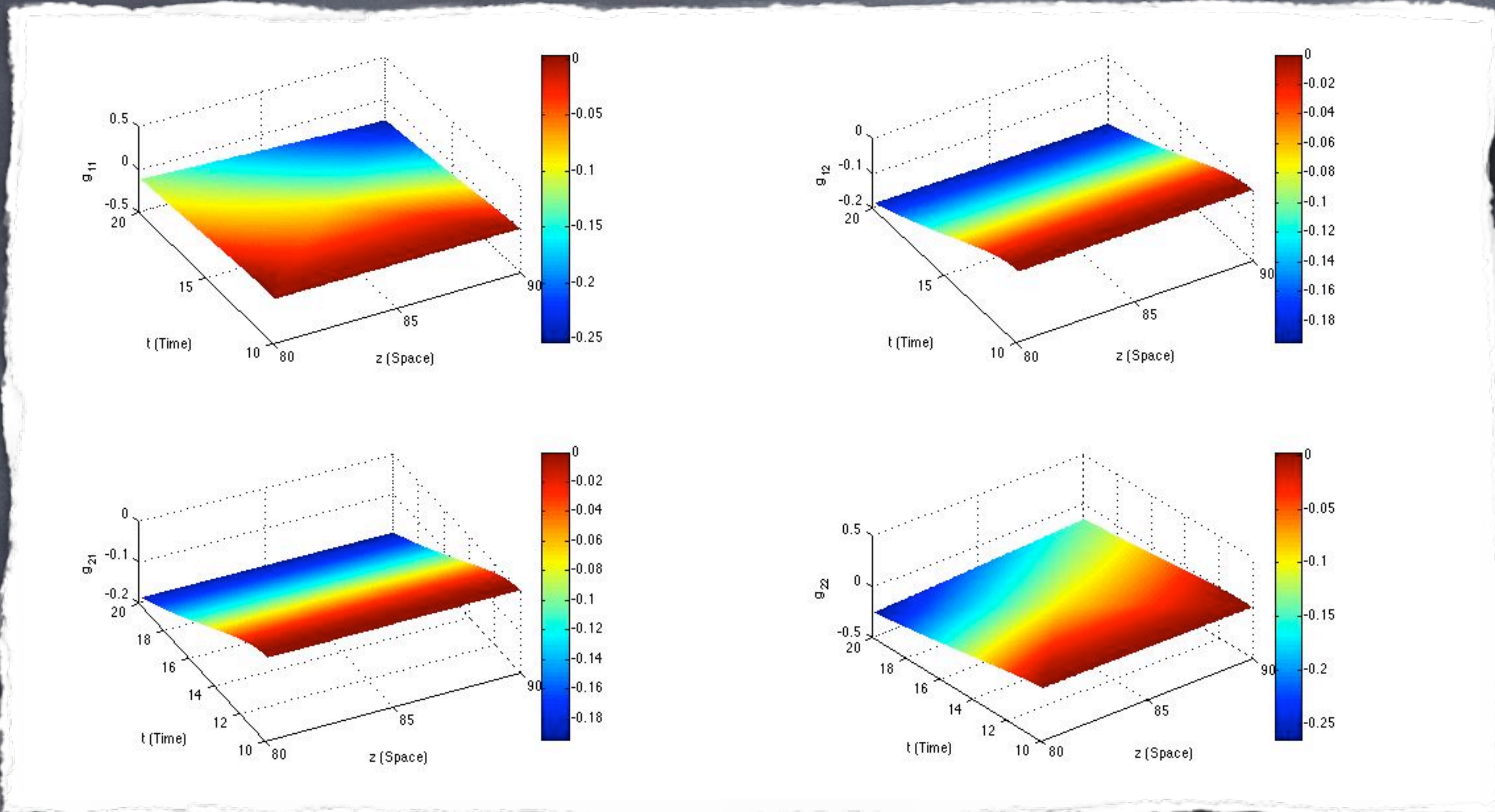
The error (in %)

What's next?

- Simulate the full Einstein field equation.
- Check the stability of the solution that was obtained by Belinskii and Zakharov.

Fin

Thank you!



The error (in %)