THE INVERSE SCATTERING TRANSFORM

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1. Motivation

Given a linear equation

$$\begin{cases} \frac{\partial u}{\partial t} & = C_0(t)\, u + C_1(t)\, \frac{\partial u}{\partial x} + \cdots + C_n(t)\, \frac{\partial^n u}{\partial x^n} \\ u(x,0) & \text{given} \end{cases} \tag{1.1}$$

we can solve the problem using the Fourier transform. This is the (linear!) integral transformation

$$u(x,t) \mapsto \mathcal{F}[u](k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx} dx$$

which turns equation 1.1 into an (infinite) set of ODEs

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} & = C_0(t)\, \hat{u} + C_1(t)\,(-ik)\, \hat{u} + \cdots + C_n(t)\,(-ik)^n\, \hat{u} \\ \hat{u}(k,0) & \text{given} \end{cases}$$

which can be solved, and then transformed back to $u(x,t)$ using the inverse Fourier transform $\mathcal{F}^{-1}$. In other words, we solve linear PDEs using the following scheme

$$u(x,0) \xrightarrow[\mathcal{F}]{I} \hat{u}(k,0) \xrightarrow[\text{solution of ODE}]{II} \hat{u}(k,t) \xrightarrow[\mathcal{F}^{-1}]{III} u(x,t)$$

However, once the PDE is not linear, the Fourier transform is in general useless since we don't know how to compute the Fourier transform of a nonlinear combination of unknown functions. As an example, the universal (i.e. appears all over the place in application!) PDE called the Korteweg-de Vreis equation (KdV equation)

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.2}$$

cannot be solved using the Fourier transform.

The Inverse Scattering Transform (IST) is a nonlinear analog of the Fourier transform which allows to transform a nonlinear PDE into an (infinite) set of ODEs, thus allowing to solve many nonlinear PDEs! Since the PDEs we will start from are nonlinear, we cannot expect the IST to be a linear transformation (and it is not linear...). The amazing thing about it is that the transformation is based on a purely physical argument (in Scattering Theory) that apriori has nothing to do with the initial PDE we started from.

2. IST for the KdV

2.1. **Introduction.** In (quantum) scattering theory, we study the collison of a wave-particle represented by a wave function $\psi(x,t)$ and a potential $u(x,t)$. (PLACE PICTURE). The common problems are separated into two branches:

- The Direct Scattering Problem: We know the potential $u(x,t)$ and want to understand the evolution of the wave function $\psi(x,t)$.
- The Inverse Scattering Problem: This covers the more realistic problem in which we shoot a beam from one side, watch it coming out from the other side (i.e. we know $\psi(x,t)$) and then we want to understand the potential $u(x,t)$ that caused the change in the wave-function. The fact that we know to solve the inverse scattering problem is the reason most medical imaging system work (like MRI)!

We will soon see that if we let $u(x,t)$ in the KdV Eq. 1.2 be the potential, we can find how the potential evolves (namely the solution of the KdV!) by considering the direct and inverse scattering problems. This will give us the solution of the KdV <u>without actually solving the KdV equation</u> using the following scheme

$$
\begin{array}{ccc}
u(x,0) & \dashrightarrow & u(x,t) \\
I\downarrow & & \uparrow III \\
s(0) & \xrightarrow{II} & s(t)
\end{array}
$$

where

- I : Solving the direct scattering problem, namely calculating the initial scattering data $s(0)$ from the initial condition $u(x,0)$ by solving the Schrödinger equation with potential $u(x,0)$.
- II : Evolving the scattering data. We will solve this step in the general case.
- III : Solving the inverse scattering problem, namely constructing the potential $u(x,t)$ based on the scattering data $s(t)$.

2.2. **Scattering Theory.** Consider the one-dimensional Schrödinger operator $L = -\frac{d^2}{dx^2} + u(x)$, and assume $u(x)$ is sufficiently smooth and vanishes as $|x| \to \infty$. We study the eigenvalue problem

$$L\psi = -\frac{d^2\psi}{dx^2} + u(x)\psi = \lambda\psi$$

for ψ that are bounded on the real axis.

2.3. **Integration of the KdV Equation.** We consider the Schrödinger equation

$$L\psi = -\frac{d^2\psi}{dx^2} + u(x,t)\psi = \lambda\psi \tag{2.1}$$

where the potential u also depends on time t. The dependence of u on t means that the wave function ψ and the eigenvalue λ will also depend on t. Peter Lax noticed that if we define the operator A as

$$A = 4\frac{d^3}{dx^3} - 3\left(u\frac{d}{dx} + \frac{d}{dx}u\right)$$

then the equation

$$\dot{L} = [L, A] \tag{2.2}$$

is equivalent to KdV Eq. 1.2 (think about it: $\dot{L}$ is simplify the operator of multiplication by u_t, while by simplifying the right-hand side, you'll see it is just multiplication by $6uu_x - u_{xxx}$). The operators L and A are called the Lax pair for the KdV equation, as they can be used to give a different representation for the KdV.

Theorem 1. $\dot{L} = [L, A]$ *for* A *skew-symmetric (in* $L^2(\mathbb{R})$*) if and only if* $L(t)$ *is unitarily equivalent to* $L(0)$.

Proof. ($\Leftarrow$) $L(t) = U(t)\,L(0)\,U^{-1}(t)$ for a unitary operator $U(t)$. Differentiating with respect to time t, we have

$$\begin{aligned}
\dot{L} &= \dot{U}L(0)\,U^{-1} - UL(0)\left(U^{-1}\dot{U}U^{-1}\right) \\
&= \dot{U}\left(U^{-1}U\right)L(0)\,U^{-1} - \left(UL(0)\,U^{-1}\right)\dot{U}U^{-1} \\
&= \left(\dot{U}U^{-1}\right)\left(UL(0)\,U^{-1}\right) - \left(UL(0)\,U^{-1}\right)\left(\dot{U}U^{-1}\right) \\
&= \left(\dot{U}U^{-1}\right)L - L\left(\dot{U}U^{-1}\right) \\
&= \left[L, -\dot{U}U^{-1}\right]
\end{aligned}$$

so $\dot{L} = [L, A]$ for $A = -\dot{U}U^{-1}$. A is skew-symmetric since

$$\begin{aligned}
A^* &= -\left(\dot{U}U^{-1}\right)^* \\
&= -\left(U^{-1}\right)^*\dot{U}^* \\
&= -U\dot{U^{-1}} \\
&= -U\left(-U^{-1}\dot{U}U^{-1}\right) \\
&= \dot{U}U^{-1} \\
&= -A
\end{aligned}$$

($\Rightarrow$) By the previous computation already expect to have $A = -\dot{U}U^{-1}$. But since A is independent of t it is clear that defining

$$U(t) = \exp(-At)$$

satisfies $A = \dot{U}U^{-1}$ (U is also unitary since $U^*U = \exp(-A^*t)\exp(-At) = \exp(At)\exp(-At) = 0$). But then,

$$\begin{aligned}
\frac{d}{dt}\left(U^{-1}LU\right) &= -U^{-1}\dot{U}U^{-1}LU + U^{-1}\dot{L}U + U^{-1}L\dot{U} \\
&= U^{-1}ALU + U^{-1}(LA - AL)\,U + U^{-1}L\dot{U} \\
&= U^{-1}ALU + U^{-1}LAU - U^{-1}ALU - U^{-1}LAU \\
&= 0
\end{aligned}$$

and clearly $U^{-1}LU\ |_{t=0} = L(0)$. Therefore $U^{-1}LU = L(0)$, or $L(t)$ is unitarily equivalent to $L(0)$. $\square$

Since $L(t)$ and $L(0)$ are unitarily equivalent, they have the same spectrum, because if f is an eigenfunction of $L(0)$ with eigenvalue k^2, then

$$L(Uf) = UL(0)f = U\lambda f = \lambda(Uf)$$

and we see that Uf is an eigenfunction of $L(t)$ with the same eigenvalue. Therefore, the eigenvalues of $L(t)$ are independent of time!

We take the derivative of the Schrödinger Eq. 2.1 with respect to time, obtaining

$$\dot{L}f + L\dot{f} = k^2\dot{f}$$

and using the Lax Eq. 2.2

$$LAf - ALf + L\dot{f} - k^2\dot{f} = 0$$

or since $Lf = k^2 f$

$$\left(L - k^2\right)\left(\dot{f} - Af\right) = 0$$

i.e. $\dot{f} + Af$ is an eigenfunction of L with the same eigenvalue as for f. We saw previously that

$$f = e^{-ikx} + o(1) \text{ as } x \to -\infty$$

and this asymptotic behavior is independent of time. Therefore by the definition of A

$$\dot{f} + Af = 4ik^3 e^{-ikx} + o(1) \text{ as } x \to -\infty$$

but by the definition of $f = e^{-ikx}$ as $x \to -\infty$ we see that

$$\dot{f} + Af = 4ik^3 f + o(1) \text{ as } x \to -\infty$$

Remember that as $x \to \infty$ we defined

$$\varphi = a(k,t)\, e^{-ikx} + b(k,t)\, e^{ikx} + o(1) \text{ as } x \to \infty$$

therefore as $x \to \infty$ (recalling that u vanishes) we have

$$\dot{a}e^{-ikx} + \dot{b}e^{ikx} = \left(-4\frac{d^3}{dx^3} + 4ik^3\right)\left(ae^{-ikx} + be^{ikx}\right) \text{ as } x \to \infty$$

or

$$\begin{aligned} \dot{a} &= 0 \\ \dot{b} &= 8ik^3 b \end{aligned} \tag{2.3}$$

this is called the Gardner-Greene, Kruskal and Miura (GGKM) equation. It means that the time evolution of the scattering data for the continuous spectrum is given by a trivial set of ordinary differential equation! In the discrete case, since $i\mathcal{X}_n$ are the zeros of the time-invariant $\dot{a} = 0$, we have $\mathcal{X}_n = 0$. The time-dependence of $b_n(t)$ is the coefficient in the asymptotic expansion

$$\varphi(x, i\mathcal{X}_n) = b_n(t)\, e^{-\mathcal{X}_n x} + o\left(e^{-\mathcal{X}_n x}\right) \text{ as } x \to \infty$$

which gives (similarly to what we did before)

$$\dot{b_n} = 8\mathcal{X}_n^3 b_n$$

Finally, we may write the time evoluation of the scattering data as

$$s(t) = \left\{ r(k,0)\, e^{8ik^3 t}; \mathcal{X}_n, b_n e^{8\mathcal{X}_n^3 t}, n = 1, \ldots, N \right\}$$

References

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[2] A. C. Newell, "Solitons in Mathematics and Physics", 1985