Complex functions and integral transformations
Session 4

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Harmonic Conjugates

Recall that $f = u + iv$ is analytic at a point $z_0 = x_0 + iy_0$ if and only if u, v satisfy the Cauchy-Riemann equations

$$u_x = v_y \qquad u_y = -v_x \tag{1}$$

and $u, v \in C^1$ in a neighborhood of (x_0, y_0).

Definition 1 *We say $u : D \to \mathbb{R}$ is an harmonic function whenever $u \in C^2(D)$ and u satisfies Laplace's equation*

$$\triangle u = 0$$

. $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian operator.

Definition 2 *Let u be an harmonic function in a domain D. We say v is harmonic conjugate to u in D if v is harmonic and satisfies the Cauchy-Riemann equations together with u. This relation is not symmetric (due to the minus sign in the Cauchy-Riemann equations).*

Theorem 3 *$f = u + iv$ is analytic in a domain $D \iff u, v$ are harmonic conjugate in D.*

Theorem 4 *In a simply-connected domain D, every harmonic function u has a harmonic conjugate function v.*

Exercise 1 *Is there an analytic function f such that $Re(f) = x^2 y$? What is it?*

Define $u = x^2 y$, and notice its Laplacian is $u_{xx} + u_{yy} = 2y \neq 0$ so such f does not exist.

Exercise 2 *Find an analytic function f such that $Re(f) = x^3 - 3xy^2$.*

$u = x^3 - 3xy^2$ is indeed harmonic. We search for an harmonic conjugate function v. By the first Cauchy-Riemann equations

$$v_y = u_x = 3x^2 - 3y^2$$

therefore

$$v(x, y) = 3x^2 y - y^3 + C(x)$$

The second Cauchy-Riemann equation gives

$$6xy + C'(x) = v_x = -u_y = 6xy$$

Therefore $C'(x) = 0$ and $C(x) = const$. This means $v(x, y) = 3x^2 y - y^2 + const$ is the harmonic conjugate of $u(x, y) = x^3 - 3xy^2$. Therefore

$$f(x + iy) = x^3 - 3xy^2 + i(3x^2 y - y^3 + const) = z^3 + const$$

is an entire function.

Exercise 3 *Prove that if u, v are C^2 functions then*

$$\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} f = \frac{1}{4} \triangle f$$

We compute,

$$\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} f = \frac{1}{4}(\partial_x + i\partial_y)(\partial_x - i\partial_y) f = \frac{1}{4}(f_{xx} - i f_{xy} + i f_{yx} + f_{yy}) = \frac{1}{4} \triangle f$$

Exercise 4 *Let $u(x, y)$ be an harmonic function.*

- *Prove that $f(x + iy) = \frac{\partial}{\partial z} u(x, y) = \frac{1}{2} u_x - \frac{i}{2} u_y$ is analytic.*
- *If u_x is harmonic, find its harmonic conjugate.*

For the first part, it is sufficient to prove that $\frac{1}{2} u_x$ and $-\frac{1}{2} u_y$ satisfy the Cauchy-Riemann equations and are C^1. u is harmonic $\implies u \in C^2 \implies u_x, u_y \in C^1$. The first Cauchy-Riemann equation is

$$\partial_x \frac{1}{2} u_x = \frac{1}{2} u_{xx} = -\frac{1}{2} u_{yy} = \partial_y \left(-\frac{1}{2} u_y \right)$$

and is clearly satisfied since u is harmonic. A similar argument works for the second equation. Therefore f is analytic.

For the second part, we already showed f is analytic. Therefore $g = 2f = u_x - i u_y$ is analytic and $-u_y$ is the harmonic conjugate of u_x.

Exercise 5 *Let $f : \mathbb{C} \to \mathbb{R}$ be a C^2 and harmonic function in a domain D. Show that $F(z) = \frac{\partial f}{\partial z}$ is analytic in D.*

F is defined to be

$$F(z) = \frac{1}{2}(f_x - if_y)$$

f is real $\implies$ its partials are real and if we denote $F = u + iv$ then

$$u = \frac{1}{2}f_x \qquad v = -\frac{1}{2}f_y$$

It is clear that they are C^1 as $f \in C^2$. All that is left to show is that u, v satisfy the Cauchy-Riemann equations. Notice that

$$u_x = \frac{1}{2}f_{xx} = -\frac{1}{2}f_{yy} = v_y$$

since f is harmonic and

$$u_y = \frac{1}{2}f_{xy} = \frac{1}{2}f_{yx} = -v_x$$

since f is C^2 (so partials commute). Therefore F is analytic in D.

Elementary Functions

The exponential function

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) \tag{2}$$

The trigonometric functions

$$\begin{aligned} \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\ \tan(z) &= \frac{\sin z}{\cos z} \\ \cot(z) &= \frac{\cos z}{\sin z} \end{aligned} \tag{3}$$

The hyperbolic functions

$$\begin{aligned} \cosh(z) &= \frac{e^{z} + e^{-z}}{2} \\ \sinh(z) &= \frac{e^{z} - e^{-z}}{2} \\ \tanh(z) &= \frac{\sinh z}{\cosh z} \\ \coth(z) &= \frac{\cosh z}{\sinh z} \end{aligned} \tag{4}$$

Exercise 6 *Prove the identity* $\cosh^2 z - \sinh^2 z = 1$.

Just use their definition.

Exercise 7 *Prove* $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$.

Starting from the right-hand side,

$$\begin{aligned} \cos(z)\cos(w) - \sin(z)\sin(w) &= \frac{1}{4}\left(e^{i(z+w)} + e^{i(z-w)} + e^{-i(z-w)} + e^{-i(z+w)}\right) \\ &\quad -\frac{1}{4i^2}\left(e^{i(z+w)} - e^{i(z-w)} - e^{-i(z-w)} + e^{-i(z+w)}\right) \\ &= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} \\ &= \cos(z+w) \end{aligned} \tag{5}$$

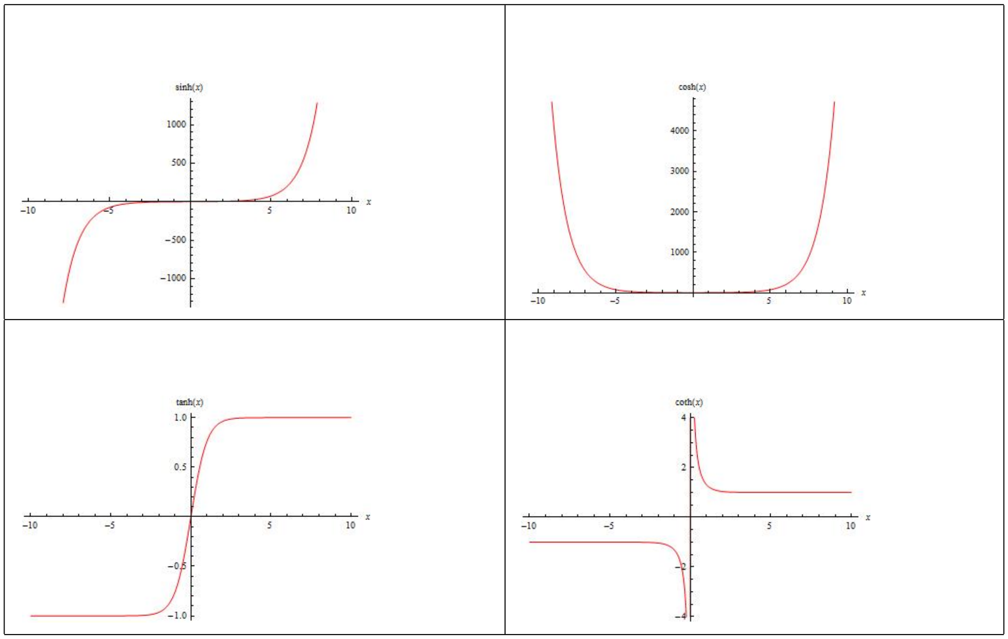

Figure 1: The hyperbolic functions on the real line.

Exercise 8 *Compute the derivatives of* $\sin z, \cos z, \sinh z, \cosh z$.

Exercise 9 *Write each of the functions* $\sin z, \cos z, \sinh z, \cosh z$ *as* $u + iv$.

$$\begin{aligned}
\sin(x+iy) &= \frac{e^{i(x+iy)} - e^{-i(x-iy)}}{2i} \qquad (6)\\
&= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}\\
&= \cos x\frac{e^{-y}-e^{y}}{2i} + i\sin x\frac{e^{-y}+e^{y}}{2i}\\
&= \sin x\cosh y + i\cos x\sinh y
\end{aligned}$$

One can compute the other functions similarly to obtain

$$\begin{aligned}
\sin(x+iy) &= \sin x\cosh y + i\cos x\sinh y \qquad (7)\\
\cos(x+iy) &= \cos x\cosh y - i\sin x\sinh y\\
\sinh(x+iy) &= \sinh x\cos y + i\cosh x\sin y\\
\cosh(x+iy) &= \cosh x\cos y + i\sinh x\sin y
\end{aligned}$$

Exercise 10 *Write the following rational functions using partial fraction decomposition:*

1. $f_1 = \frac{z+3}{z^2-3z-40}$
2. $f_2 = \frac{10z^2+12z+20}{z^3-8}$
3. $f_3 = \frac{10z^2-63z+29}{z^3-11z^2+40z-48}$
4. $f_4 = \frac{z+1}{(z+2)(z^2+1)^5}$

First remember that on the complex plane the fundamental theorem of algebra holds - namely, every polynomial of order ≥ 1 has all of its roots in the plane. This means that every polynomial can be reduced to a product of linear functions.

For the first function

$$f_1 = \frac{z+3}{z^2-3z-40} = \frac{z+3}{(z-8)(z+5)}$$

So we expect,

$$f_1 = \frac{A}{z-8} + \frac{B}{z+5}$$

Multiplying by $z-8$ and setting $z=8$ gives $A = \frac{z+3}{z+5} |_{z=8} = \frac{11}{13}$. Multiplying by $z+5$ and setting $z=-5$ gives $B = \frac{z+3}{z-8} |_{z=-5} = \frac{2}{13}$. So

$$f_1 = \frac{11/13}{z-8} + \frac{2/13}{z+5}$$

For the second function the denominator is only partially reducible,

$$f_2 = \frac{10z^2+12z+20}{(z-2)(z^2+2z+4)}$$

So we write

$$f_2 = \frac{A}{z-2} + \frac{Bz+C}{z^2+2z+4}$$

Use the same trick to find the constants. Multiply by $z-2$ and set $z=2$ gives $A = \frac{10z^2+12z+20}{(z^2+2z+4)} |_{z=2} = 7$. Plugging $z=0$ gives $C=4$. We get B by multiplying by z and taking $z \to \infty$, so that $B+A=10$, namely $B=3$. Note we can decompose the denominator to two linear functions using complex numbers.

For the third function, the denominator reduces to linear factors with multiplicity so

$$f_3 = \frac{10z^2-63z+29}{(z-3)(z-4)^2} = \frac{A}{z-3} + \frac{B}{z-4} + \frac{C}{(z-4)^2}$$

Multiplying by $z-3$ and setting $z=3$ gives $A=-70$. Multiplying by $(z-4)^2$ and setting $z=4$ gives $C=-63$. Multiplying by z and taking $z\to\infty$ gives $A+B=10$ so $B=80$.

For the last function

$$f_4=\frac{z+1}{z+2}(z^2+1)^5=\frac{A}{z+1}+\frac{B_1z+C_1}{z^2+1}+\cdots+\frac{B_5z+C_5}{(z^2+1)^5}$$

and we compute the coefficients similarly. We can also write each z^2+1 as $(z+i)(z-i)$ and reduce them.

A riddle

Build four identical triangles from six identical rods. You are not allowed to modify the rods in anyway, and must use all of them.