Complex functions and integral transformations
Session 3: Cauchy-Riemann Equations

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Theorem 1 *$f = u + iv$ is differentiable at $z_0 = x_0 + iy_0 \iff u, v$ are differentiable at $(x_0, y_0$ and satisfy the Cauchy-Riemann equations*

$$u_x = v_y \qquad u_y = -v_x \tag{1}$$

Exercise 1 *Let f be differentiable in a domain D such that $|f(z)| = const$ for all $z \in D$. Prove that f is constant in D.*
Geometrically, this means that if a differentiable function maps a domain to a circle, then it actually maps it to a single point (circle of radius 0).

If $|f(z)| \equiv 0$ then $f(z) \equiv 0$ for all $z \in D$. Otherwise $|f(z)| = c \neq 0$ and $c \in \mathbb{R}$. This means $|f(z)|^2 = u^2 + v^2 = c^2$. Differentiating w.r.t x and y gives

$$\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

Our assumption that $u^2 + v^2 \neq 0$ means $\begin{bmatrix} u \\ v \end{bmatrix} \neq \vec{0}$. This means we just found a non-trivial solution to the linear equation! Therefore the determinant of the matrix must be zero, namely

$$\det \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} = 0 \tag{3}$$

or, $u_x v_y - u_y v_x = 0$. Since f is differentiable u, v satisfy the Cauchy-Riemann equations, therefore $(u_x)^2 + (u_y)^2 = (v_x)^2 + (v_y)^2 = 0$. So $u = \text{const}$ and $v = \text{const}$, meaning that f is constant in D.

Exercise 2 *Prove that if f is analytic in $\mathbb{C}$ ($\equiv$ entire) and $(Re(f))^2 = Im(f)$ for all $z \in \mathbb{Z}$ then $f = const$ in $\mathbb{C}$.*

The assumption gives $u^2 = v$. f is analytic $\implies$ u, v satisfy the Cauchy-Riemann equations, yielding

$$u_x = v_y = 2uu_y = 2u(-2uu_x) = -4u^2 u_x \tag{4}$$

therefore $u_x(1 + 4u^2) = 0$, and $u_x = 0$. Similarly $u_y = 0$ and we may conclude u is constant. Because $v = u^2$ it must be constant as well. Hence, $f =$ constant.

Exercise 3 *Let $f(z) = u(x, y) + iv(x, y)$ is analytic at $z = 0$ then $g(z) = \overline{f(\bar{z})}$ at $z = 0$.*

Note that

$$g(x + iy) = \overline{f(x - iy)} = u(x, -y) - iv(x, -y) \equiv \tilde{u} + i\tilde{v} \tag{5}$$

u, v are differentiable about $(0, 0)$ $\implies$ $\tilde{u}, \tilde{v}$ are differentiable about $(0, 0)$. We need to check they satisfy the Cauchy-Riemann equations as well:

$$\begin{aligned} \tilde{u}_x &= u_x \\ \tilde{u}_y &= -u_y \\ \tilde{v}_x &= -v_x \\ \tilde{v}_y &= v_y \end{aligned} \tag{6}$$

f is analytic about $z = 0$ implies that there exists a radius $r > 0$ such that u, v satisfy the Cauchy-Riemann equations in $D = \{z : |z| \le r\}$. If $z \in D$ then $\overline{z} \in D$ and the Cauchy-Riemann equations are satisfied for $\tilde{u}, \tilde{v}$. Therefore g is analytic at $z = 0$.

Definition 2 *The $z/\bar{z}$ (Wirtinger) derivatives are defined to be:*

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{aligned} \tag{7}$$

The operator $\frac{\partial}{\partial z}$ is not sufficient to differentiation by z (the $z/\bar{z}$ operators are always defined if $u, v \in C^1$, but recall that sometimes $u, v \in C^1$ but $f'(z)$ still doesn't exist).

Exercise 4 *Prove that if f is differentiable at z_0 then $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.*

Easy,

$$\frac{\partial f}{\partial z} = \frac{1}{2}(f_x - if_y) = \frac{1}{2}\left[u_x + iv_x - i(u_y + iv_y)\right] = u_x + iv_x \tag{8}$$

where the last equality follows from the Cauchy-Riemann equations. But this is simply $f'(z_0)$.

Exercise 5 *Prove that f satisfies the Cauchy-Riemann equations if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.*

Compute,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[u_x + iu_y + i(v_x + iv_y)\right] = \frac{1}{2}\left[(u_x - v_y) + i(u_y + v_x)\right] \tag{9}$$

showing that it vanishes if and only if the Cauchy-Riemann equations hold.

Exercise 6 *Prove that in polar coordinates the Cauchy-Riemann equations are*

$$u_r = \frac{1}{r}v_\theta \qquad v_r = -\frac{1}{r}u_\theta \tag{10}$$

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Notice that

$$\begin{aligned} \partial_r &= x_r\partial_x + y_r\partial_y = cos\theta\partial_x + \sin\theta\partial_y \\ \partial_\theta &= x_\theta\partial_x + y_\theta\partial_y = -r\sin\theta\partial_x + r\cos\theta\partial_y \end{aligned} \tag{11}$$

From the Cartesian Cauchy-Riemann equations

$$\begin{aligned} u_r &= \cos\theta u_x + \sin\theta u_y = \cos\theta v_y - \sin\theta v_x \\ v_r &= \cos\theta v_x + \sin\theta v_y = -\cos\theta u_y + \sin\theta u_x \\ u_\theta &= -r\sin\theta u_x + r\cos\theta u_y \\ v_\theta &= -r\sin\theta v_x + r\cos\theta v_y \end{aligned} \tag{12}$$

giving $u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$ as we wanted.

Remark 3 *One can prove that if $f(z)$ is differentiable then in polar coordinates*

$$f'(z) = u_x + iv_x = (\cos\theta - i\sin\theta)(u_r + iv_r) \tag{13}$$

A riddle

Using 6 sticks with identical sizes, create 4 equilateral triangles without having modifying the sticks (or having any extra parts popping out).