Complex functions and integral transformations
Session 1: Introduction

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Complex numbers

- The complex field $\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$ where $i^2 = -1$.

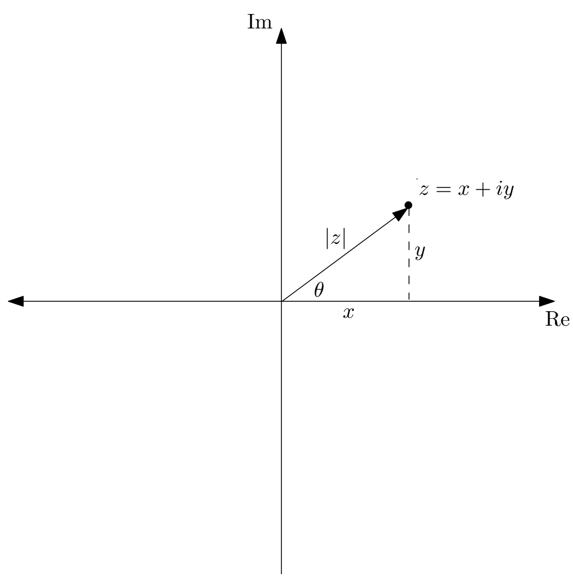

Figure 1: A number in the complex plane

- The Cartesian representation of a complex number $z = x + iy$.
- For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:
  - Addition / difference: $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$.

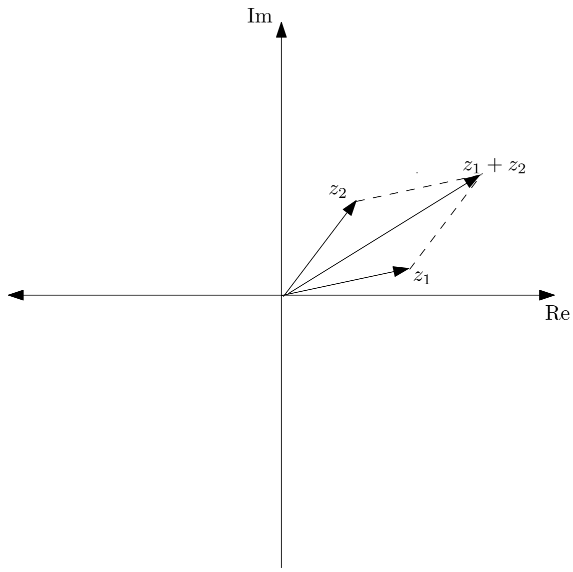

Figure 2: Addition of complex numbers

- Product:

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1) \tag{1}$$

- The complex conjugate of z is $\bar{z} = \overline{x + iy} = x - iy$, it satisfies:
  - $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
  - $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

- The quotient of two numbers z_1, z_2 is:

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(x_1 + iy_1) \cdot (x_2 - iy_2)}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \tag{2}$$

- The real part: $Re(z) = x = \frac{z+\bar{z}}{2}$.
- The imaginary part: $Im(z) = y = \frac{z-\bar{z}}{2i}$.
- The modulus $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$, satisfying
  - $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
  - $||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$ (Triangle inequality, see figure 2 for its geometric interpretation)

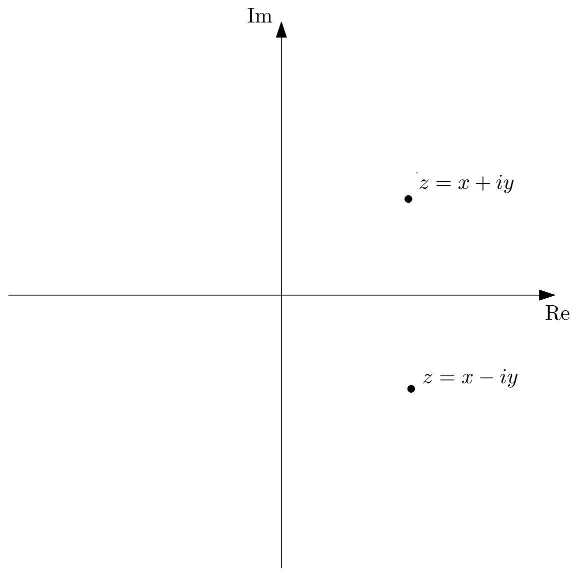

Figure 3: The complex conjugate

- The argument: $tan\theta = \frac{y}{x}$ (note that this *does not* mean $\theta = \arctan\frac{y}{x}$!)

  Notation: $\theta = \arg(z)$. The argument is *a multi-valued function*, satisfying

  - $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$
  - $\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$

  Note that both of this formulas are *set equalities*.

- The principal argument: $Arg(z) \in \arg(z)$ such that $Arg(z) \in (-\pi, \pi)$. This is a single-valued function. Some people defined it to be $Arg(z) \in (\alpha, \alpha + 2\pi)$.

- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

  It leads to one of the most beautiful equations in mathematics: $e^{i\pi} + 1 = 0$, connecting the five most important numbers.

- Polar representation of a complex number: $z = re^{i\theta}$ with $r = |z|$ and $\theta \in \arg(z)$. See figure 1 for its geometric interpretation.

- De moivre's formula: for all $n \in \mathbb{Z}$ $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (follows from $(e^{i\theta}) = e^{in\theta}$). Therefore for any complex numbers

$$z^n = (re^{i\theta})^n = r^n(\cos n\theta + i\sin n\theta) \tag{3}$$

- The n-th root: $\sqrt[n]{z} = \sqrt[n]{re^i\theta} = \sqrt[n]{r}(\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right))$ for $k = 0, 1, \ldots, n-1$.

Problems

1. Write down the quotient $\frac{(1+i)^9}{(1-i)^7}$ in the Cartesian form $z = x + iy$.
2. Find the set of points z in the complex plane satisfying the inequality
$$|z - i| - |z + i| < 2 \tag{4}$$
3. Compute $\arg(z)$ and $Arg(z)$ for:
   (a) $z = -6 - 6i$
   (b) $z = -\pi$
   (c) $z = 10i$
   (d) $z = \sqrt{3} - i$
4. Prove that
   (a) $e^{z+i\pi} = -e^z$
   (b) $\overline{e^z} = e^{\bar{z}}$
5. Prove the parallelogram equality:
$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \tag{5}$$
   for all $z, w \in \mathbb{C}$
6. Find all the solutions of $(z - 1)^3 = -3$.

   Solution:
$$z = 1 + \sqrt[3]{-3} = 1 + \sqrt[3]{3}\sqrt[3]{e^{i\pi}} = 1 + \sqrt[3]{3}e^{i\frac{\pi+2\pi k}{3}} \tag{6}$$
   where $k = 0, 1, 2$.
7. Prove that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.

   Solution: Use Euler's formula for $e^{i3\theta} = (e^{i\theta})^3$ and take the imaginary part.
8. Plot the domains:
   (a) $A = \{z \in \mathbb{C} | z \neq 0, 0 \leq Re(\frac{1}{z})\}$

   Solution: $Re(\frac{1}{z}) = Re(\frac{\bar{z}}{|z|^2}) = \frac{x}{x^2+y^2}$ so it is clear that $Re(\frac{1}{z}) \geq 0$ iff $x > 0$. So $A = \{z \in \mathbb{C} | Re(z) \geq 0, z \neq 0\}$.

(b) $B = \{z \in \mathbb{C} | z \neq 0, 1 \leq Re(\frac{1}{z})\}$

Solution: Similarly to before, an algebraic analysis shows B is the closed circle of radius 1/2 around $z = \frac{1}{2}$ excluding the point $z = 0$.

(c) $C = \{z \in \mathbb{C} | \frac{\pi}{4} \leq Arg(z + i) \leq \frac{\pi}{2}\}$

A riddle

Take the L-shaped region 1 in figure 4 and split it into two identical parts. Now split region 2 into three identical parts. Last but not least, split region 3 into four identical parts.

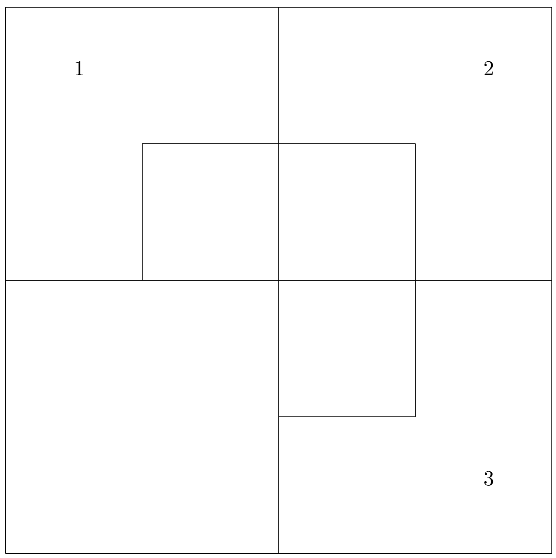

Figure 4: The complex conjugate