Black holes and Solitons

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Based on the work

'Integration of the Einstein equations by Means of the Inverse Scattering Problem Technique and Construction of Exact Soliton Solutions'
by V. A. Belinski and V. E. Zakharov

The Gist of the Inverse Scattering Transform (IST)

The Fourier Transform

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{linear PDE} \arrow[r, "\mathcal{F}"] \arrow[d, dashed] & \text{ODE} \arrow[d] \\
\text{solution} & \mathcal{F}[\text{solution}] \arrow[l, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

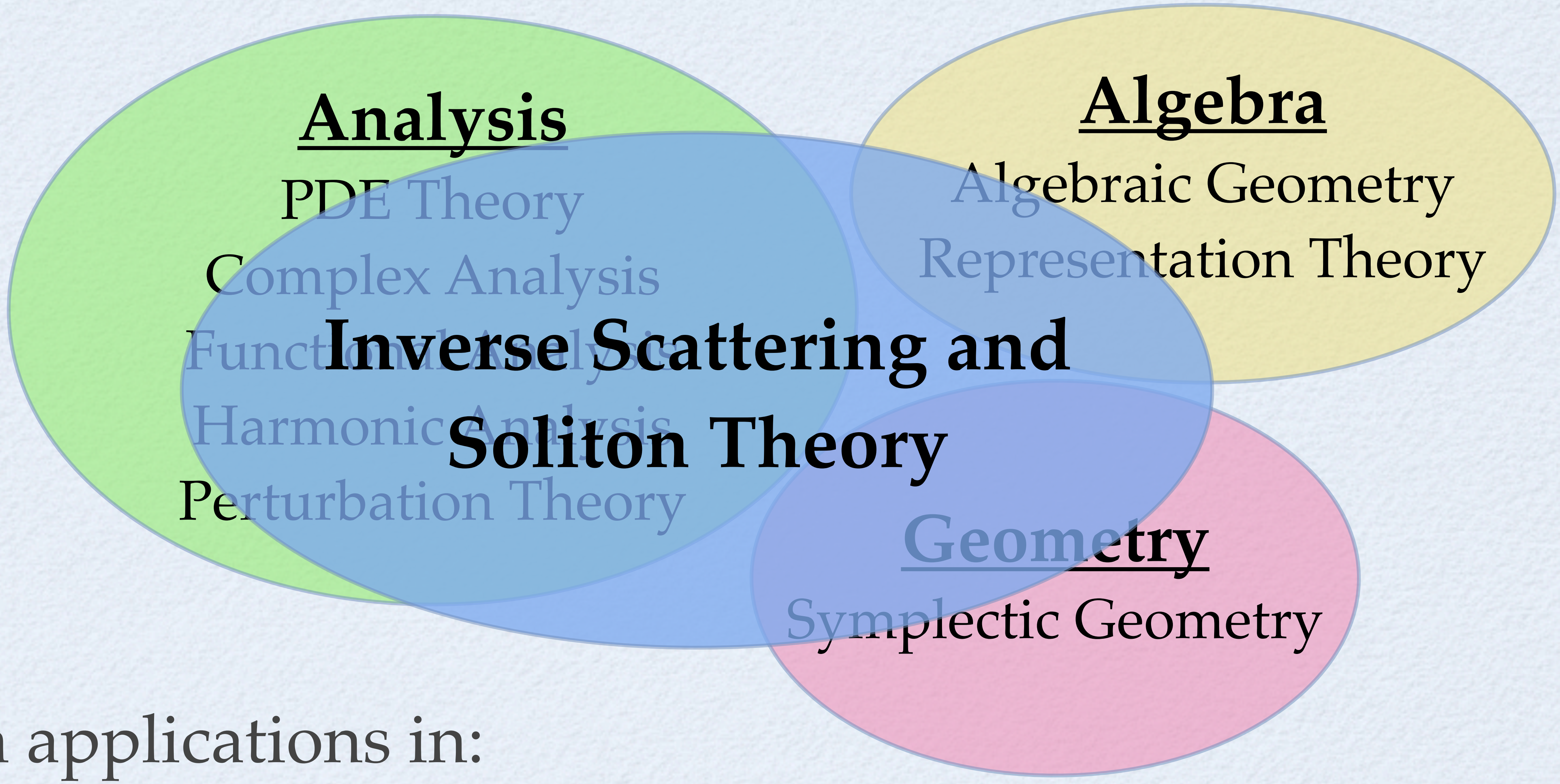
The Inverse Scattering Transform

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{nonlinear PDE} \arrow[r, "\text{direct transform}"'] \arrow[d, dashed] & \text{2 linear PDEs} \arrow[d] \\
\text{solution} & \text{solution} \arrow[l, "\text{inverse transform}"']
\end{tikzcd}$$

Get Solitons!

Example #1
Example #2
Example #3

The IST with respect to other fields of math



with applications in:

- hydrodynamics
- optics
- plasma physics
- quantum field theory
- string theory
- solid state physics
- general relativity
- and more...

Special Relativity

Postulate: The speed of light c is the same for all observers.

$$c(dt) = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Or equivalently $-c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = 0$

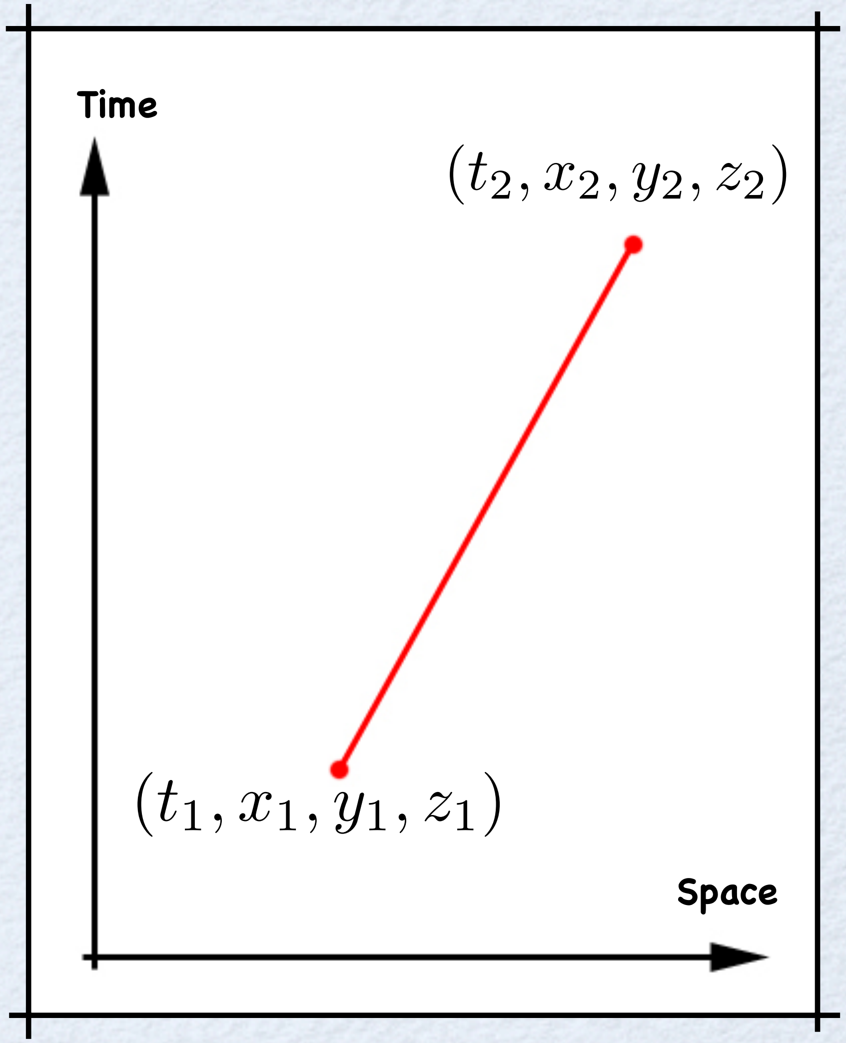
Introduce the notation $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$
and the matrix:

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

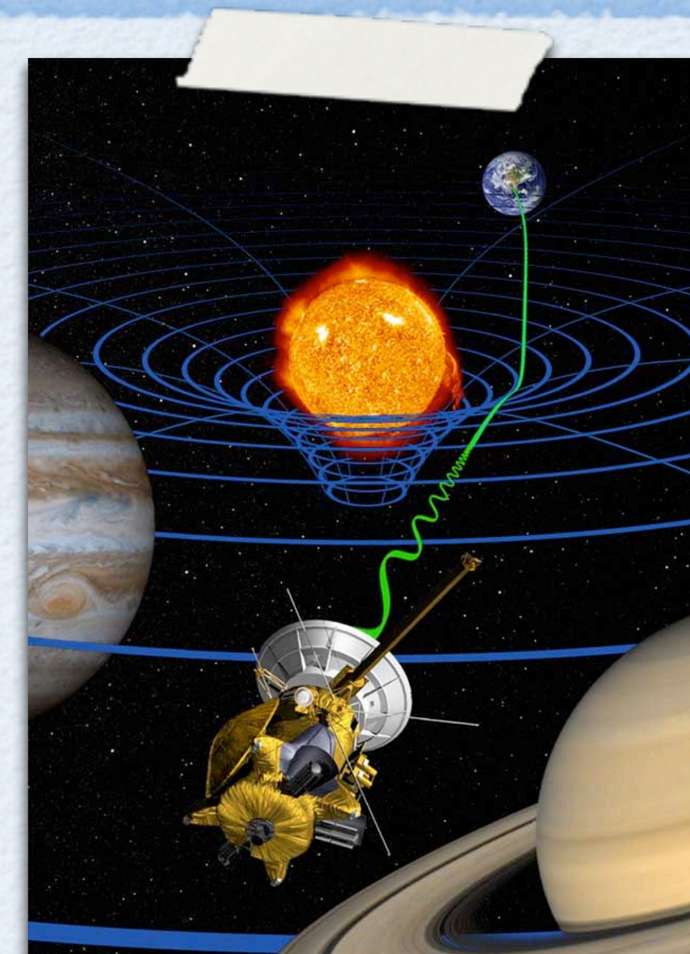
and we can write the condition as:

$$-ds^2 := \eta_{\alpha\beta} dx^\alpha dx^\beta = 0$$

(sum over repeated indices)



General Relativity



<u>The Equivalence Principle</u>: gravity affects all bodies in the same way, independently of the body's composition.

The spacetime interval: $-ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$

Spacetime curves in the presence of matter according to Einstein’s equation:

$$\underbrace{R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R}_{\text{“curvature”}} = \cancel{(8\pi G)\underbrace{T_{\alpha\beta}}_{\text{“matter (energy)”}}}$$

In the absence of matter we get Einstein’s vacuum equation:

$$R_{\alpha\beta} = 0$$

Our Setup

The spacetime metric in matrix form:

$$g_{\alpha\beta} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$$

$$f = f(t, z)$$
$$g_{ab} = g_{ab}(t, z)$$

We get the spacetime interval:

$$-ds^2 = f(-dt^2 + dz^2) + g_{ab}dx^a dx^b \quad (a, b = 1, 2)$$

Einstein's vacuum equation:

$$R_{\mu\nu} = 0$$

$$R_{ab} = 0$$

$$R_{00} + R_{33} = 0$$
$$R_{03} = 0$$

$$\det(g) = \alpha^2$$

$$\underbrace{(\alpha g_{,t} g^{-1})}_{-A}{}_{,t} - \underbrace{(\alpha g_{,z} g^{-1})}_{B}{}_{,z} = 0$$

Easy equation for f

The IST for Einstein's Eq

1

Take a particular "seed" solution g_0

2

Solve two **linear** PDEs

$$\begin{aligned} D_1\psi_0 &= \frac{A_0}{\lambda-\alpha}\psi_0 \\ D_2\psi_0 &= \frac{B_0}{\lambda+\alpha}\psi_0 \end{aligned} \Longrightarrow$$

Get $\psi_0(t,z,\lambda)$

3

Define

$$\chi = I + \sum_{k=1}^{n}\left(\frac{R_k}{\lambda-\mu_k} + \frac{\bar{R}_k}{\lambda-\bar{\mu}_k}\right)$$

(there are explicit formulas for R_k, μ_k in terms of ψ_0)

4

Get an n-solitonic solution!

$$g = \chi\,(0)\,g_0$$

Example

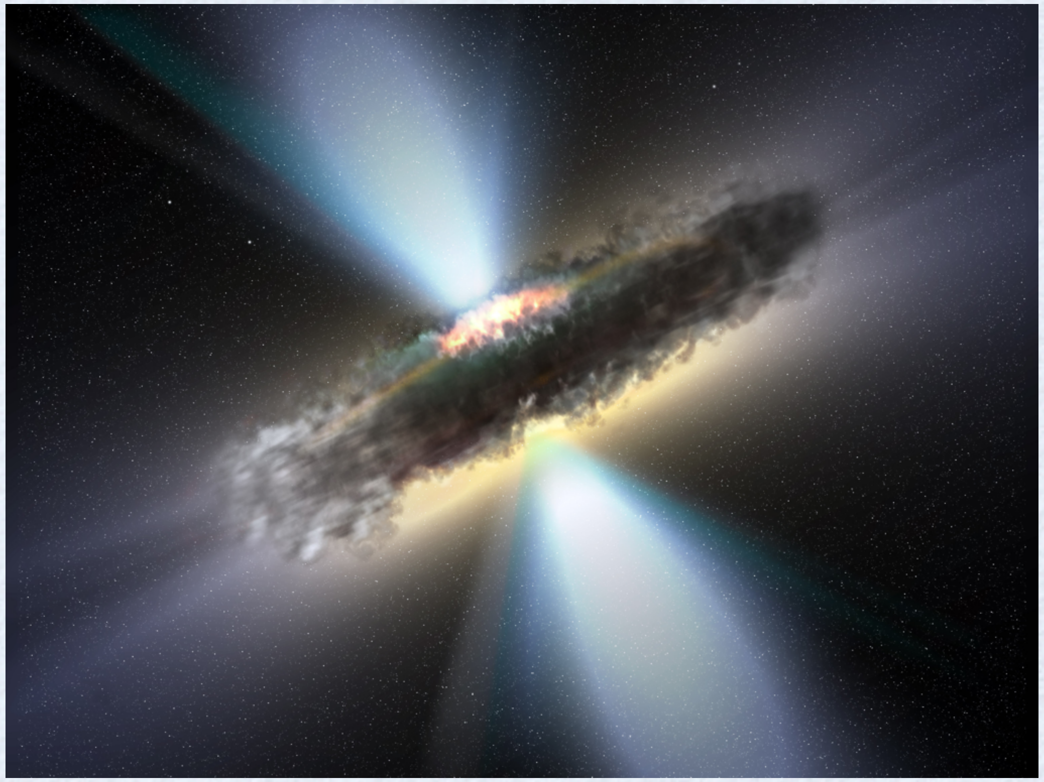
The simplest solution of Einstein's vacuum equation is...

$$-ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

or in cylindrical coordinates

$$-ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2 \qquad f_0 = 1 \quad g_0 = \begin{bmatrix} r^2 & 0 \\ 0 & 1 \end{bmatrix}$$

In this case the number of solitons has to be even.



Do the math, and get for two solitons:

The simplest application of the IST gives the Kerr solution which describes a **rotating black hole**. It is considered to be the most important solution of Einstein's equation.

My Research

- Find infinitely many conservation quantities of the Einstein equation (so far I have 5...)
- Study more seeds
- Hopefully generalize the theory for metrics that depend on three variables