Zeta Zeros and Quantum Chaos

Yaron Hadad and Jordan Schettler

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Outline

Classical Mechanics

Newtonian Mechanics

Newton's 2nd Law: The trajectory $\mathbf{x}(t)$ of a particle with mass m is determined by

$$m\ddot{\mathbf{x}} = F\left(\mathbf{x}, \dot{\mathbf{x}}\right)$$

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Usually, the force is conservative and independent of $\dot{\mathbf{x}}$, and thus can be written as (minus) the gradient of a function:

$$m\ddot{\mathbf{x}} = -\frac{\partial V}{\partial \mathbf{x}}$$

where $V(\mathbf{x})$ is called the potential function.

Lagrangian Mechanics

In Calculus of Variations, we study the ‘action’ functional

$$S[\mathbf{x}] = \int_{t_1}^{t_2} L(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

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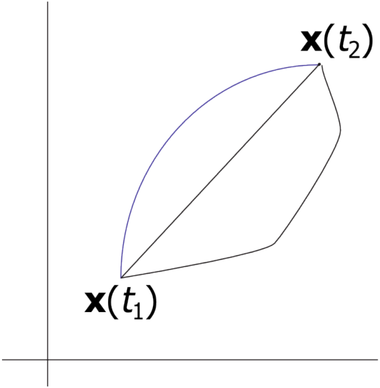
where L is called the Lagrangian. This is a functional over the space of all possible continuous paths.



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The path extremizing $S[\mathbf{x}]$ satisfies Euler-Lagrange's equation:

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Taking $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})$ gives back Newton's 2nd Law.

Instead, Lagrange preferred to think of Nature as optimal: we can think of the particle as traveling in such a way that it always minimizes (extremizes) the action S.

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These are the celebrated *Hamilton's equations*. Consequence:

$$\frac{dH}{dt} = 0 \text{ (conservation of energy)}$$

Riemann’s 1859 Paper

Euler Product Formula

Theorem (Dirichlet, 1838)

For $s > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

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Note: The divergence at $s = 1$ implies the infinitude of primes.

The Zeta Function

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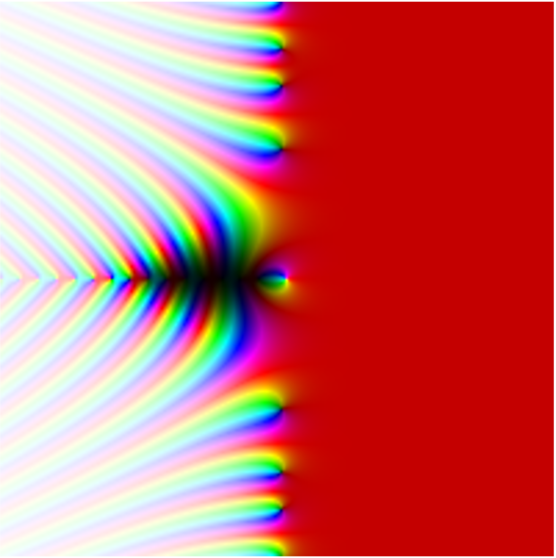
Riemann used complex analysis to find a formula for the number of primes $\pi(x)$ less than a given magnitude x.

Color Map of $f(s) = s : \Re(s), \Im(s) \in [-30, 30]$

Color Map of $\zeta(s) : \Re(s), \Im(s) \in [-30, 30]$



The Functional Equation and Special Values

Riemann also established the identity

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We have $\zeta(0) = -1/2$ and for $n = 1, 2, \cdots$

$$\zeta(-2n) = 0 \text{ (trivial zeros)} \qquad \zeta(1-2n) = -\frac{B_{2n}}{2n}$$

$$\zeta(2n) = \frac{(2\pi)^{2n}|B_{2n}|}{2(2n)!} \qquad \zeta(1+2n) = \ ???$$

where $\dfrac{x}{e^x - 1} = \displaystyle\sum_{m=0}^{\infty} B_m \frac{x^m}{m!}$.

The "Completed" Zeta Function $Z(s)$

The function

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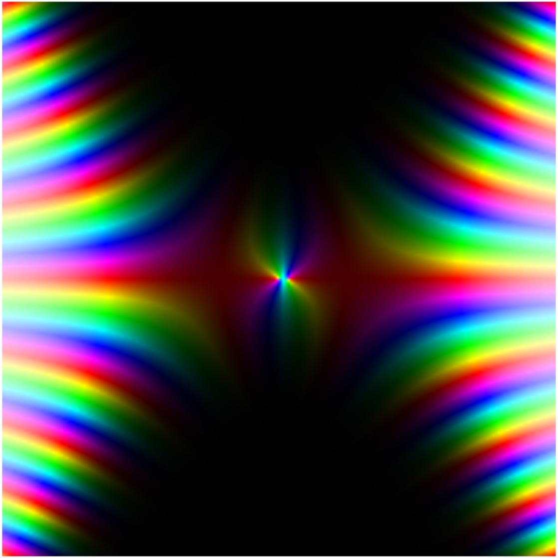
The functional equation becomes

$$Z(s) = Z(1-s)$$

Color Map of $Z(s)$: $\Re(s), \Im(s) \in [-30, 30]$



Hadamard Product Formula

Theorem (Hadamard, 1893)

$$s(s-1)Z(s) = \prod_{\alpha}\left(1 - \frac{s}{\alpha}\right)$$

where a nontrivial zero α of $\zeta(s)$ is paired with its "twin" $1-\alpha$.

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We can combine the Hadamard and Euler product formulas to count primes with nontrivial zeros or vice versa.

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where $\tilde{\pi}(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$.

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Fourier inversion gives

$$\tilde{\pi}(x) = -\frac{1}{2\pi i} \cdot \frac{1}{\log(x)} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left(\frac{\log(\zeta(s))}{s}\right) x^s\, ds \qquad (a > 1)$$

Counting Primes with Zeros

We get

$$\tilde{\pi}(x) = \mathrm{Li}(x) - \sum_{\alpha} \mathrm{Li}(x^{\alpha}) - \log(2) + \int_x^{\infty} \frac{dt}{(t^3 - t)\log(t)}$$

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Möbius inversion gives

$$\begin{aligned} \pi(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \tilde{\pi}(x^{1/n}) \\ &= \langle \pi(x) \rangle + \pi_{\mathrm{osc}}(x) \end{aligned}$$

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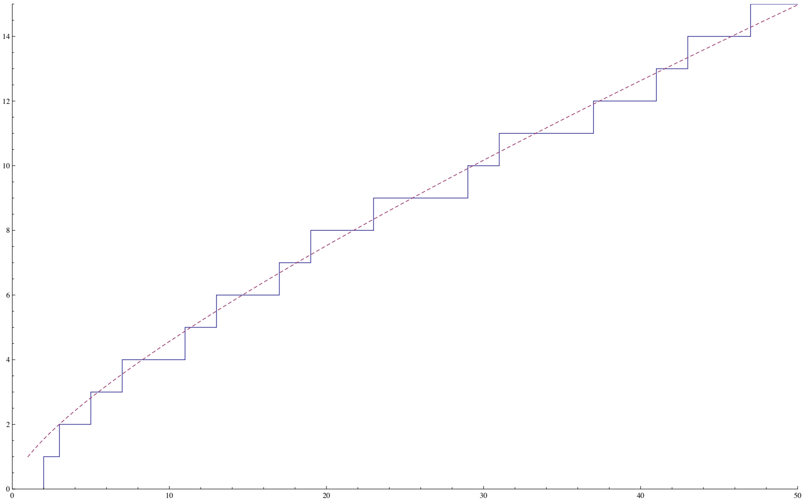
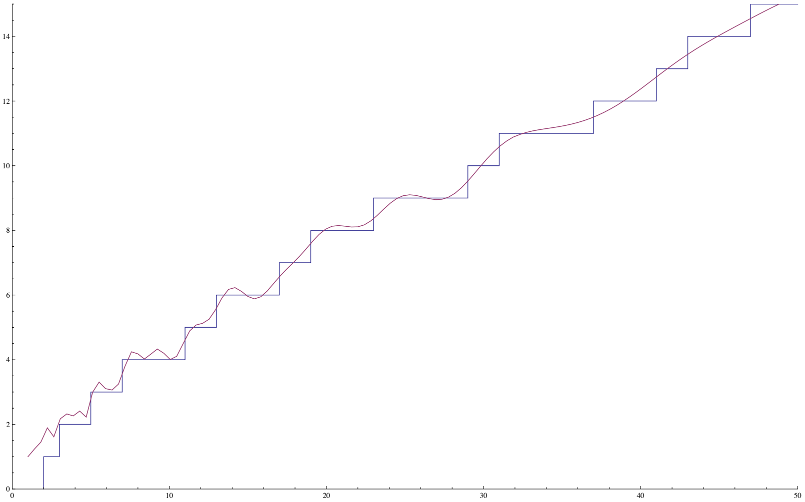
PNT = Prime Number Theorem (Hadamard and de la Vallée-Poussin, 1896): $\nexists$nontrivial zero α on the line $\Re(s) = 1$.

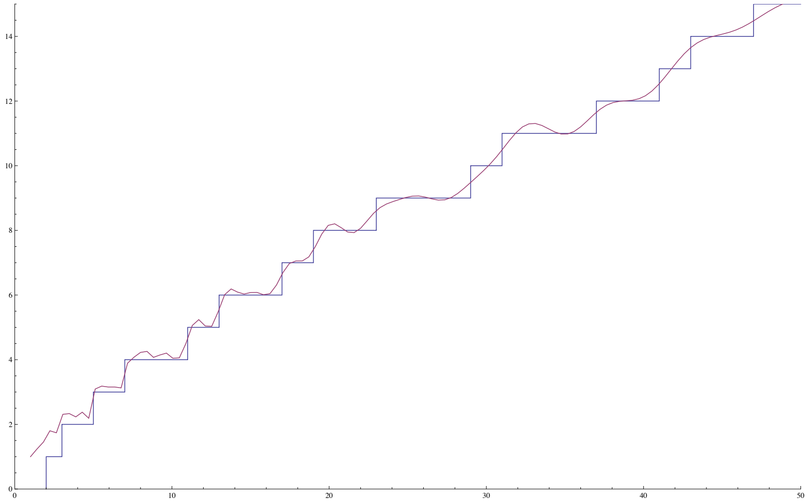
$\pi(x)$ Approximated with $\langle\pi(x)\rangle$



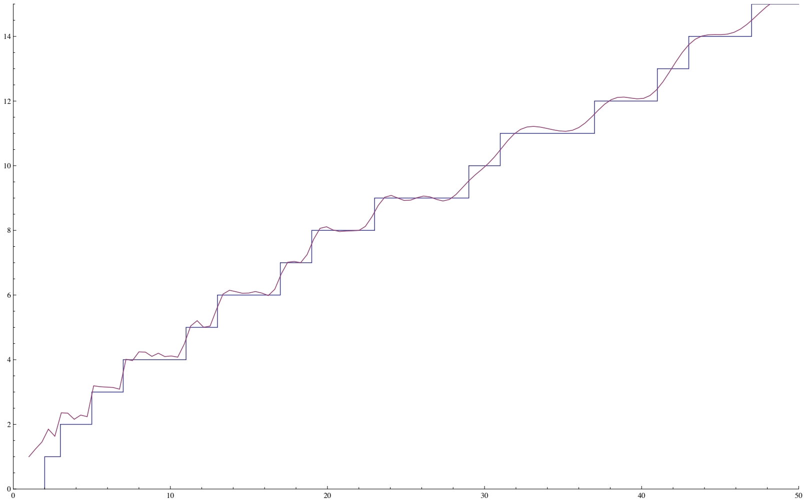
$\pi(x)$ Approximated with $\langle\pi(x)\rangle$ and 5 Pairs of Zeros



$\pi(x)$ Approximated with $\langle\pi(x)\rangle$ and 10 Pairs of Zeros



$\pi(x)$ Approximated with $\langle\pi(x)\rangle$ and 15 Pairs of Zeros



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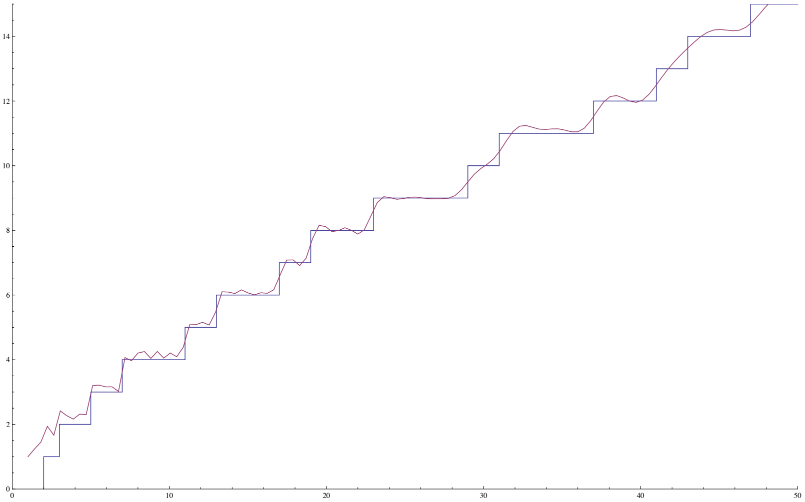
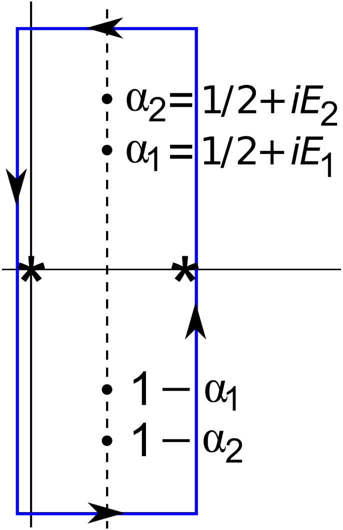


$R : \Re(s) \in [-\epsilon, 1+\epsilon], \Im(s) \in [-E, E]$



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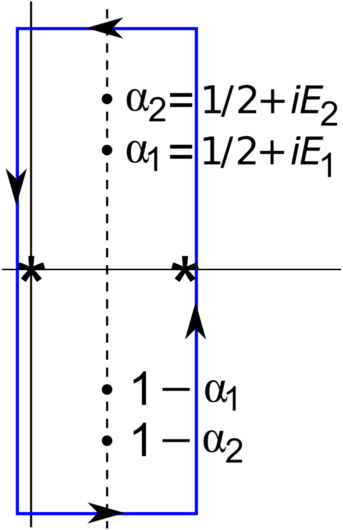

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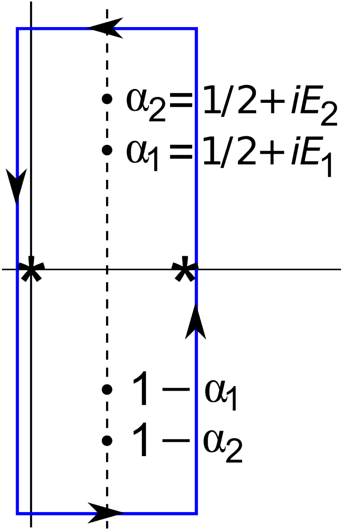

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$$= \frac{4}{2\pi} \Im \int_{1+\varepsilon}^{\frac{1}{2}+iE} \frac{Z'(s)}{Z(s)}\, ds$$

$$= \frac{2}{\pi} \Im \log(Z(1/2 + iE))$$

Counting Zeros with Primes

$$N(E) = 1 + \frac{1}{\pi}\Im \log \pi^{-\frac{1}{4} - i\frac{E}{2}} + \frac{1}{\pi}\Im \log \left(-\frac{3}{4} + i\frac{E}{2}\right)! + \frac{1}{\pi}\Im \log \zeta\left(\frac{1}{2} + iE\right)$$

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$$\text{``}=\text{''} \underbrace{\frac{E}{2\pi}\left(\log\frac{E}{2\pi}-1\right) + \frac{7}{8} + \cdots}_{\langle N(E)\rangle} + \underbrace{\frac{-1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m}\frac{\sin(mET_p)}{e^{(m/2)\lambda_p}}}_{N_{\mathrm{osc}}(E)}$$

Here $T_p = \lambda_p = \log(p)$

Quantum Mechanics (QM)

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   For example the momentum $\mathbf{p}$ has the corresponding operator $\hat{\mathbf{p}} = -i\hbar\frac{d}{d\mathbf{x}}$

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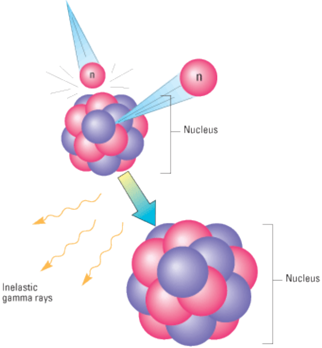

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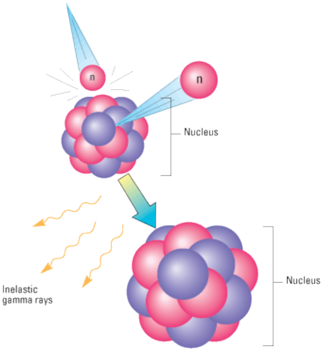

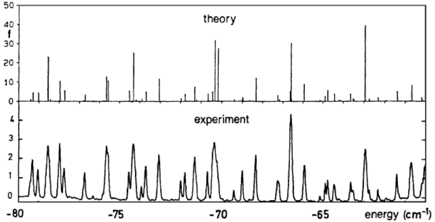

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Its Fourier transform ($t \to E$) is the Green function $G^{+}(\mathbf{x},\mathbf{x}';E)$.

Göttingen (1912-1914)

"[Landau] asked me one day: 'You know some physics. Do you know a physical reason that the Riemann hypothesis (RH) should be true?'... I answered, if the nontrivial zeros... were so connected with the physical problem that the RH would be equivalent to the fact that all the eigenvalues of the physical problem are real." —George Pólya

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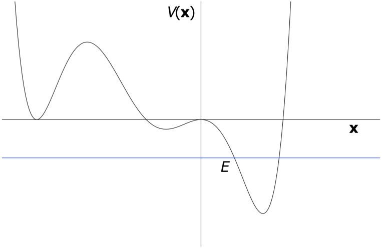
This would imply the RH! Hilbert also had this idea.

The Riemann Operator

Unstable Equilibriums



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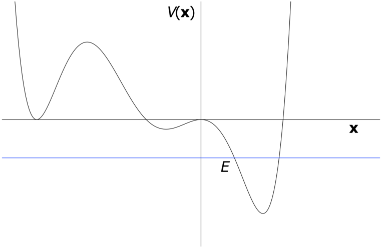

Nearby an equilibrium point $x_0 = 0$, the potential is

$$V(x) = V(0) + V'(0)\mathbf{x} + \frac{1}{2}V''(0)\mathbf{x}^2 + \dots$$

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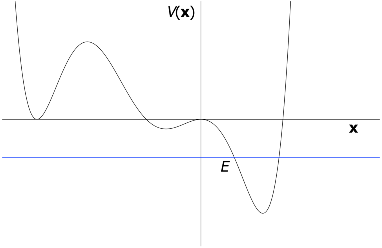

If the equilibrium point is unstable, the Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \frac{1}{2}K\mathbf{x}^2 \quad (K > 0)$$

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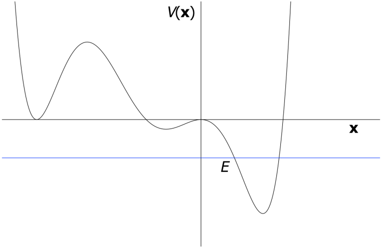

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$$H = \frac{\mathbf{p}^2}{2m} - \frac{1}{2}K\mathbf{x}^2 \quad (K > 0)$$

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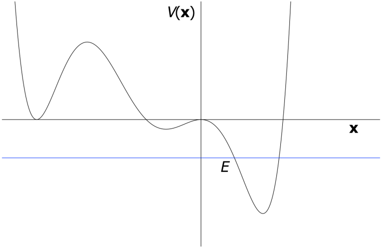

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Namely, this Hamiltonian describes the general motion of a particle about an unstable equilibrium point.

Energy Levels of the Riemann Operator

For any (classically bound) Hamiltonian $H(\mathbf{x}, \mathbf{p})$,

$$N(E) = \text{"\# of energy levels } E_n \leq E\text{"} = A\left(E\right)/2\pi\hbar + O\left(1\right)$$

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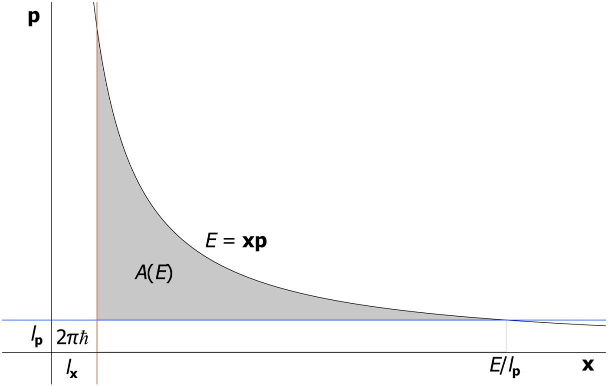

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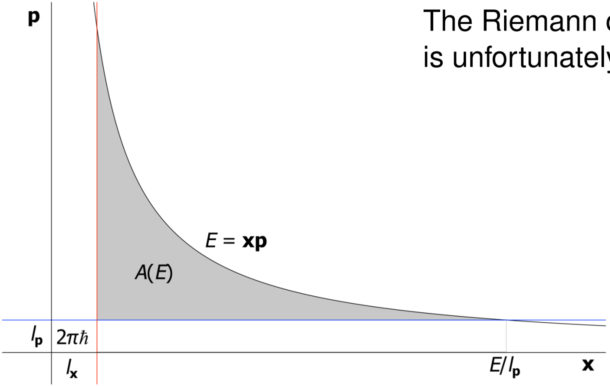

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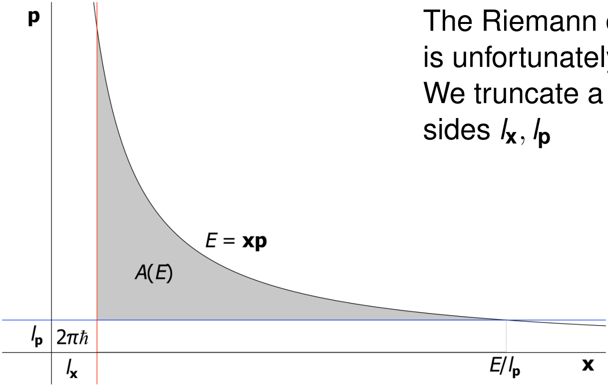

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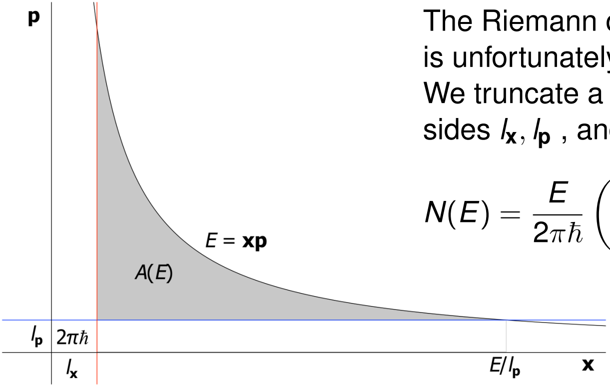

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$$N(E) = \frac{E}{2\pi\hbar}\left(\log\frac{E}{2\pi\hbar} - 1\right) + \cdots$$

The Oscillatory Part of the Energy Levels

Note:

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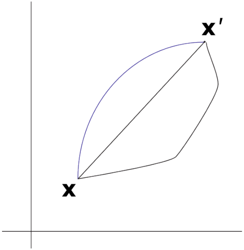
This means that we can approximate the energy levels density function by integrating the Green function!

Feynman Path Integrals

In the Lagrangian formulation a particle takes a *unique* path that minimizes the action S.



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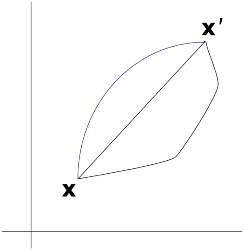

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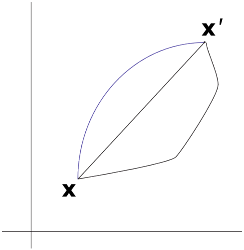

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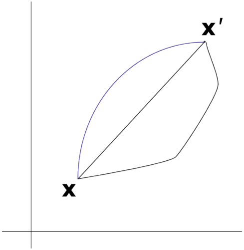

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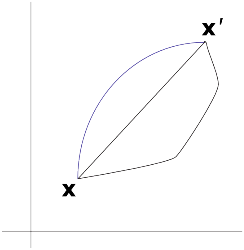

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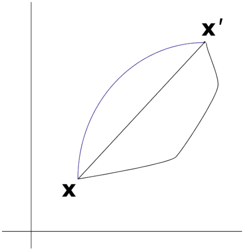

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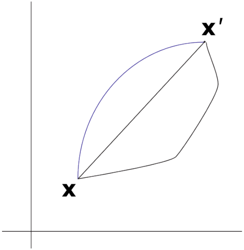

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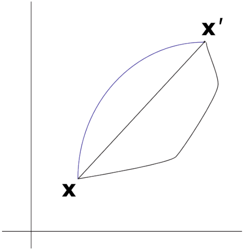

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More Evidence?

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$\exists$functions $r, \theta \colon \mathbb{R} \to \mathbb{R}$ s.t. $\zeta(1/2 + iE) = r(E)e^{-i\theta(E)}$

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Theorem (Montgomery, 1972)

If the Fourier transform $\hat{f}$ is C^∞ and supported in $(-1, 1)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m,n\leq N} f(\phi_m - \phi_n) = f(0) + \int_{-\infty}^{\infty} f(x)K(x)\,dx$$

where $K(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$. Conj: we can take $f = 1_{[\alpha,\beta]}$.

A Chance Encounter in 1972

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Dyson noticed that these patterns for Riemann zeros were the same as those predicted by quantum physicists for energy levels in the nucleus of heavy atoms via random matrix theory.

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Let $A \in U(N)$ ($N \times N$ matrices : $A\overline{A}^T = 1$) w/ eigenvalues $e^{i\theta_n}$.

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Theorem (Dyson, 1963)

If $f(x) \to 0$ as $x \to \pm\infty$, then

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\sum_{m,n\leq N} f(\phi_m - \phi_n)\right] = f(0) + \int_{-\infty}^{\infty} f(x)K(x)\,dx$$

where $K(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$. In fact, we can take $f = 1_{[\alpha,\beta]}$.

Thank You!