INTRODUCTION TO LAGRANGIAN MECHANICS

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1. Newtonian Mechanics

According to Newton's laws of motions, which were first published on July 5, 1687 in the book "Philosophiae Naturalis Principa Mathematica", space is the 3-dimensional Euclidean space $\mathbb{R}^3$, in which the motion of a body is governed by Newton's second law[1]: The rate of change of the momentum of a body is proportional to the resultant force acting on the body and is in the same direction. If we represent the position of the body as a function of time by a vector $\vec{x}(t)$, its velocity as $\vec{v}(t) = \dot{\vec{x}}(t)$, its acceleration as $\vec{a}(t) = \ddot{\vec{x}}(t)$ and its mass is constant in time, we may express Newton's second law symbolically by:

$$\boxed{\vec{F} = m\vec{a}}$$

where $\vec{F}$ is the total force exerted on the body and m is the mass of the body. This is a second-order differential equation for the position of the body $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^3$, which has a unique solution given the position $\vec{x}(t_0)$ and the velocity $\dot{\vec{x}}(t_0)$ of the body at a given time t_0 (assuming that the force $\vec{F}$ is a nice-enough function, e.g. $\vec{F} \in C^1$.)

The work done by the force $\vec{F}$ when the body is going from point 1 to point 2 is defined to be:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{l}$$

and thus for a constant mass m (which will be assumed from now on),

$$\begin{aligned} W_{12} &= \int_1^2 \vec{F} \cdot d\vec{l} \\ &= m \int_1^2 \frac{d\vec{v}}{dt} \cdot \vec{v} dt \\ &= \frac{m}{2} \int_1^2 \frac{d}{dt} \left(\vec{v}^2\right) dt \\ &= \frac{m\vec{v}_2^2}{2} - \frac{m\vec{v}_1^2}{2} \end{aligned}$$

[1] An equivalent way of stating Newton's second law, is using Newton's principle of determinacy: "The initial state of a mechanical system (namely, the position and velocity at some moment of time) uniquely determines all of its motion". In particular, the initial positions and velocities determine the acceleration. In other words, there is a function $\vec{f}$ such that $\ddot{x} = \vec{f}(x, \dot{x}, t)$. Here the function $\vec{f}$ corresponds to the force exerted on the body divided by the mass of the body.

The <u>scalar</u> quantity $\frac{m\vec{v}^2}{2}$ is called the *kinetic energy* of the body and is denoted by T, so that the work done is equal to the change in the kinegtic energy:

$$W_{12} = T_2 - T_1$$

If the force field $\vec{F}$ is conservative, i.e. is independent of the path in which the body travels from the point 1 to the point 2, vector calculus tells us that $\nabla \times \vec{F} = 0$. But by Helmholtz's decomposition theorem $\nabla \times \vec{F} = 0$ if and only if $\vec{F}$ is the gradient of some scalar function of position V (remember that we assume $\vec{F} \in C^1$). Thus in such a case $\vec{F} = -\nabla V(x)$, and V is called the *potential*, or *potential energy* of the force F. The sign is taken so that the potential energy of a stone is larger if the stone is higher off the ground. The reason the force field $\vec{F}$ (and the system) is called conservative in such a case, is that we have,

$$\begin{aligned} T_2 - T_1 &= W_{12} \\ &= \int_1^2 \vec{F} \cdot d\vec{l} \\ &= -\int_1^2 \nabla V \cdot d\vec{l} \\ &= V_1 - V_2 \end{aligned}$$

Therefore if we define a new scalar quantity, $E = T + V$, called the *total energy* of the body, E is conserved and

$$T_1 + V_1 = T_2 + V_2$$

or

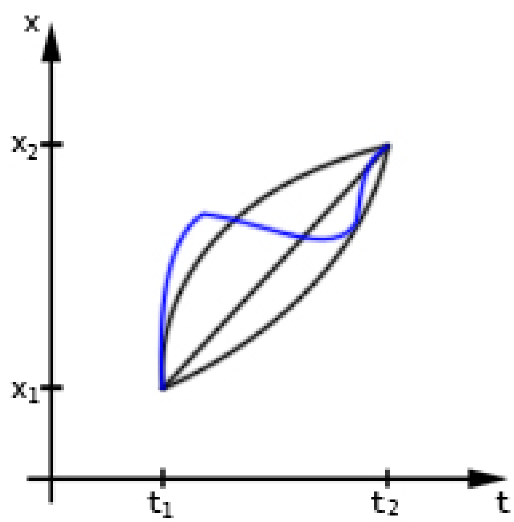
$$\frac{m\vec{v}_1^2}{2} + V_1 = \frac{m\vec{v}_2^2}{2} + V_2$$

This is the energy conservation theorem for a body (or 'the law of conservation of energy' if you wish).

2. Enter Lagrange

2.1. **Lagrangian Mechanics[2].** About a century later, in 1788, the mathematician and physicist Joseph Lagrange published a book called "Mécanique Analytique", in which he tried to reveal the true nature of that physical law in a more fundamental context. Lagrange believed that there must be some inherent beauty in the structure of things, and Nature, according to Lagrange, should be optimal. We will follow the same philosophical ideas that led Lagrange, Euler and Hamilton to the formulation of Lagrangian mechanics. For simplicity (and since it's quite difficult to draw >4 dimensional spaces...) we will take a system of one body traveling in a one dimensional space.

Fixing any two points $(x_1, t_1), (x_2, t_2)$ in space and time, and assuming that the body travels from the point x_1 at time t_1 to the point x_2 at time t_2, a priori, the body may take any path that connects these two points. It is evident that a body travels in one specific path, so we need a way to single out the unique path in which the body will travel from all the other possible paths.



Thus, Lagrange constructed a functional on the sets of all paths starting at (x_1, t_1) and ending at (x_2, t_2), that has the following form:

$$S[\gamma] = \int_\gamma L(x(t), \dot{x}(t), t)\, dt$$

The above functional is called the *action* of the system, and is obtained by integrating the *Lagrangian* of the system over the path γ connecting (x_1, t_1) to (x_2, t_2). Since Newtonian mechanics tells us that the evolution of a mechanical system is completely determined once we know the position and velocity of the body at a given time, we assume that the Lagrangian is only a function of the position $x(t)$, the velocity $\dot{x}(t)$ and the time t, though it is not difficult to generalize this idea to include derivatives of $x(t)$ of any order. Our goal will be to find a Lagrangian L for which the real path of the body will be an extremal of the action. First, we need to find the extremal of the action S, depending on our Lagrangian L. This leads

[2] A very thorough discussion of Lagrangian mechanics can be found in [Herbert Goldstein(2001), V. I. Arnold(1997)]

to the following theorem, which was proven in a correspondence between Euler and Lagrange in the 1750s:

Theorem 2.1. *(Euler-Lagrange Equation) If $x(t) \in C^1([t_1, t_2])$ is an extremal of S and $L \in C^1$, then for all $t \in [t_1, t_2]$ the following equation holds on $x(t)$:*

$$\boxed{\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)}$$

In order to formulate things a bit more rigorously, we need to say what we mean by "an extremal" of S. What norm do we consider for the space of all paths from t_1 to t_2? Since we assume that the path is a C^1 function (as a result of the Existence and Uniqueness theorem on Newton's second law), the most natural norm to use is the $\|\cdot\|_{C^1}$.

Proof. The proof of the Euler-Lagrange equations is a very good example to the type of arguments used in Calculus of Variations. The idea is to reduce this infinite dimensional extremum problem to a Calculus extremum problem, and then to solve it using standard Calculus techniques. So, we assume that $x(t)$ is an extremal of the action, and rewrite any C^1-path γ from (x_1, t_1) to (x_2, t_2) as $\gamma_\lambda(t) = x(t) + \lambda y(t)$ where $\lambda \in \mathbb{R}$ and $y(t)$ is a path for which $y(t_1) = y(t_2) = 0$. For any such given path $y(t)$, the real-valued function

$$f(\lambda) = S[\gamma_\lambda]$$

should have an extremum at $\lambda = 0$, since $x(t)$ is an extremal of S. Since $f \in C^1$, Fermat's theorem tells us that since $\lambda = 0$ is an extremum of f, $f'(0) = 0$. Thus, we use the chain rule[3] to compute the derivative of f:

$$\begin{aligned}
0 &= \frac{d}{d\lambda} f \\
&= \frac{d}{d\lambda} \int_{t_1}^{t_2} L(x + \lambda y, \dot{x} + \lambda \dot{y}, t)\, dt \\
&= \int_{t_1}^{t_2} \frac{d}{d\lambda} L(x + \lambda y, \dot{x} + \lambda \dot{y}, t)\, dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} y + \frac{\partial L}{\partial \dot{x}} \dot{y}\right) dt \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial x} y dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}} \dot{y} dt \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial x} y dt + \frac{\partial L}{\partial \dot{x}} y \,|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) y dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right) y dt
\end{aligned}$$

[3]Notice that we must assume that the position x and the velocity $\dot{x}$ are independent variables, otherwise the use of the chain rule results in a more complicated relation. We may do that, for example, by assuming that the Lagrangian L is a function of three variables $L(a, b, c)$. Thus, by writing $\frac{\partial L}{\partial x}$ we actually mean $\frac{\partial L}{\partial a}\,|_x$, and similarly $\frac{\partial L}{\partial \dot{x}}$ means $\frac{\partial L}{\partial b}\,|_{b=\dot{x}}$.

where the term $\frac{\partial L}{\partial \dot{x}} y \mid_{t_1}^{t_2}$ vanishes since by assumption $y(t_1) = y(t_2) = 0$. The equality

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \right) y dt = 0$$

holds for all paths y with $y(t_1) = y(t_2) = 0$. Since a path is continuous by definition, a necessary condition for the equality to hold is that $\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0$, namely that the Euler-Lagrange equation will be satisfied. $\square$

Even though we will soon use Euler-Lagrange equation to derive Newton's second law, it is important to note that the equation may be used to find the extremum of any functional of the form considered above! In fact, Euler and Lagrange first developed this technique in order to solve a different problem (the tautochrone problem), and only later Hamilton used it to describe general mechanical systems. Let's look at an example:

Example. The shortest path between two points in a plane

Given a C^1-function $x(t) : [t_1, t_2] \to \mathbb{R}$, the length of the graph of the function is given by:

$$S = \int_a^b \sqrt{1 + \left(\frac{dx}{dt} \right)^2} dt$$

This is an example of an extremum problem on the set of all C^1-functions defined on $[t_1, t_2]$. In this case, the Lagrangian is:

$$L(\dot{x}) = \sqrt{1 + \dot{x}^2}$$

The condition that the curve be the shortest path is that S be minimal. In this case, $\frac{\partial L}{\partial x} = 0$ and $\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}$ and Euler-Lagrange equation is:

$$0 = \frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} \right)$$

Therefore

$$\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} = c$$

where c is a constant. Thus $\dot{x} = a$ is a constant. Integrating with respect to t, we get

$$x(t) = at + b$$

which is the equation of a straight line. It is important to notice that we just proved that the straight line is an extremum path, but not a minimum! There are more advanced Calculus of Variations techniques that allow us to prove (using the notion of a second variation of a functional, which corresponds to the notion of a second derivative of a function) that this is actually a minimum.

So far we found a necessary condition that the real path of the body must satisfy in order to be an extremum of the action functional, but what is the Lagrangian L for a mechanical system? Comparing Newton's second law:

$$-\frac{\partial V}{\partial x} = \frac{d}{dt} (m\dot{x})$$

with the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right)$$

These two equations will be equivalent, if the left-hand-side and the right-hand-side of both equations coincide. Thus, we set

$$\begin{aligned}\frac{\partial L}{\partial x} &= -\frac{\partial V}{\partial x}\\ \frac{\partial L}{\partial \dot{x}} &= m\dot{x}\end{aligned}$$

(here we take the constant of integration with respect to t to be zero.) Integrating the second equation with respect to $\dot{x}$, we see that

$$L = \frac{m\dot{x}^2}{2} + c(x)$$

Substituting it back to the first equation,

$$c'(x) = -\frac{\partial V}{\partial x}$$

Therefore $c(x) = -V(x)$ (where we take the constant of integration to be zero again[4]), and the Lagrangian L is

$$L = T - V$$

Hence, by taking $L = T - V$, Euler-Lagrange equation is equivalent to Newton's second law. So a different way of stating the laws of mechanics is as follows:

Theorem 2.2. *(Hamilton's principle of stationary action[5]) The motions of a mechanical system coincide with the extremals of the action functional, where the Lagrangian of the system is the difference between the kinetic and the potential energy,* $L = T - V$.

Example 2.3. Harmonic Oscilliator

Definitely the most important example in classical physics is the harmonic oscilliator, as it is the first approximation to the motion of a particle displaced from its equilibrium position.

The potential of a 1-dimensional simple harmonic oscilliator is $V(x) = \frac{1}{2}kx^2$. The kinetic energy is $T = \frac{1}{2}m\dot{x}^2$, so that the Lagrangian is

$$\begin{aligned}L &= T - V\\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\end{aligned}$$

[4] Notice that the Lagrangian is not unique! It is unique up to adding another a solution to Euler-Lagrange's equation. We take the constants of integration to be zero because it's sufficient in that case to derive the equation of motion, and because it keeps things simple.

[5] The origin of this is principle is actually the 'principle of least action', whose historical background is rather controversial. There are several such principles: (1) Fermat's principle of least time (stated by Fermat in a letter he wrote in 1662): "The path taken between two points by a ray of light is the path that can be traversed in the least time." (2) Principle of least action (The credit is usually given to Maupertuis who stated it in 1744, but Euler stated it in the same year slightly later. Also, a copy of a letter that Leibniz sent in 1707 contains the same principle. The original letter has been lost, and even the king of Prussia entered the debate about the origin of the principle...): "The motion of a body is an extremum of the functional $\int pdq$" where p is the momentum of the body, and q is the generalized coordinate. See the section 'Lagrangian mechanics on manifolds' for more information. (3) Hamilton's principle of stationary action as we stated it (stated by Hamilton in his work "On a General Method in Dynamics" from the year 1835.) (4) D'Alembert's principle using the notion of a virtual displacment (Which is out of the scope of this paper.)

$\frac{\partial L}{\partial x} = -kx$ and $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, consequently, the Euler-Lagrange equation is

$$-kx = \frac{d}{dt}\left(m\dot{x}\right)$$

Assuming that the mass is constant with time, we have,

$$-kx = m\ddot{x}$$

which is Hooke's law. Defining $\omega = \sqrt{\frac{k}{m}}$, we have the second-order differential equation

$$\ddot{x} + \omega^2 x = 0$$

The general solution for this equation can be written as

$$x\left(t\right) = A\sin\left(wt + \phi\right)$$

where A is the amplitude, and ϕ is called the phase. Both determined by the initial conditions.

2.2. **So why does Nature extremize $T - V$?** While developing the Lagrangian mechanics, we based our treatment very strongly on Newton's second law. Actually, when we began, we set our goal to be a rederivation of Newton's second law from an optimization problem point of view. But why does nature extermize $T - V$? What is so special about it?

Remember, that the Lagrangian of the system is $T - V$ only for <u>conservative</u> forces. In such a case, we have already seen before that the total energy of the system $E = T + V$ is conserved. So quantitatively, the Lagrange of the system is also $L = E - 2V = 2T - E$. The kinetic energy T measures the amount of motion in the system, i.e. it is greater as the bodies are moving around more. On the other hand, the potential energy measures how much energy is stored in the system, that is, it measures how much could happen, but isn't happening yet (as the word 'potential' means). Thus the Lagrangian measures (in whatever way you wish to interpret it!) how much the system is active[6]: as there is more kinetic energy, the Lagrangian is greater, but as there is more potential energy, it is smaller. In most mechanical systems (but not all!) the real path of the body is not just an extremum of the action, but a <u>minimum</u> of it. Thus the system evolves in such a way that the action of the system, i.e. the total 'activity' of the system is minimal - so <u>Nature likes to be as lazy as she can!</u>[7]

[6]Notice that when we interpret the potential energy V, we actually invoke Newton's second law once more. The potential energy were defined so that we will have conservation of energy in mechanical systems, based on Newton's second law.

[7]Here we should remember again, that the Lagrangian we chose is not unique, and indeed there are many other choices for the Lagrangian function that will yield the <u>same</u> equation of motion. So in fact, we can find many other different Lagrangians that will be extremized. It seems that the fact that the Lagrangian is not unique makes the interpretation of the Lagrangian function as a measure the the 'activeness' of the system irrelevent. After all, we could have picked a different Lagrangian that gives the same equation of motions. But yet still, this Lagrangian <u>does</u> yield the 'right' equation of motion, and thus, one (out of many) possible interpretations for the evolution of a mechanical system will be the 'lazyness' of the system, as stated above.

2.3. **Lagrangian mechanics on manifolds**[8]**.** We start with some terminology. We stated before that according to Newtonian mechanics, space is the 3-dimensional manifold $\mathbb{R}^3$. If we have a system with n bodies, the position of the i^{th} body can be described by a vector function $\vec{x}^i(t) : \mathbb{R} \to \mathbb{R}^3$. The direct product of n copies of $\mathbb{R}^3$ is called the *configuration space* of the system of n bodies, and the n mappings $\vec{x}^i$ define one mapping

$$x : \mathbb{R} \to \mathbb{R}^{3n}$$

of the time axis into the configuration space, called the the *motion of the system.* Notice that the configuration space in this case is the $3n$-dimensional manifold $\mathbb{R}^{3n}$. In a more general context, when considering the motion of the system it may be necessary to take into account the constraints that limit the motion of the system. For example, the beads of an abacus are constrained to 1-dimensional motion by the supporting wires. Many other examples of constrained systems can easily be furnished. Gas molecules within a container are constrained to move inside the container by the walls of the vessel. In such a case, we classify the possible constraints into two groups. If a constraint can be expressed as equations connecting the coordinates of the bodies (and possibily the time), i.e. has the form

$$f(x,t) = 0$$

(remember that x gives us the position of <u>all</u> the bodies in the system), then the constraint is said to be *holonomic.* The constraint on the beads of an abacus discussed earlier is a holonomic constraint. If the constraint is not expressible in this fashion, it is called *nonholonomic*[9]. For example, the walls of a gas container constitute a nonholonomic constraint.

When there are no constraints on the system, we already mentioned that the configuration space is the $3n$-dimensional manifold $\mathbb{R}^{3n}$. More generally, if there are m holonomic constraints on the system, the configuration space is a $(3n-m)$-dimensional manifold M ("sitting" in an ambient manifold $\mathbb{R}^{3n}$). The dimension of the configuration space M is called the *number of degrees of freedom.* The $3n-m$ coordinates $q = \left(q^1, q^2, \ldots, q^{3n-m}\right)$ we pick to describe the motion of the system in a neighborhood of a point are called *generalized coordinate*s (god knows why people use q instead of x, but it is a rather standard notation.) A tangent vector to the configuration space is thought of as a velocity vector; its components with respect to the coordinates q are written as $\dot{q} = \left(\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{3n-m}\right)$ rather than $\left(v^1, \ldots, v^{3n-m}\right)$, and are called *generalized velocities.* Before, the Lagrangian was a function of the position, the velocity and time. Now, if the Lagrangian is independent of time (as it was in our previous example) we can simply say that the Lagrangian is a function on the tangent bundle TM of the configuration space M, namely $L : TM \to \mathbb{R}$ and $L = L(q, \dot{q})$ (remember that the tangent bundle TM has a natural manifold structure by taking the local coordinates to be $\left(q^1, \ldots, q^{3n-m}, \dot{q}^1, \ldots, \dot{q}^{3n-m}\right)$).

A quite remarkable fact about Euler-Lagrange equation is that they hold on any configuration space as long as we interpret things in the right way. Instead of

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

[8] More information on the subject can be found in [V. I. Arnold(1997), Frankel(2003)]

[9] Nonholonomic constraints will not be considered in this paper

as it was for one body moving in one dimension, we simply have to replace the position x with the generalized coordinate q^i, and the velocity $\dot{x}$ with the generalized velocity $\dot{q}^i$, and we get $(3n-m)$-equations called the Euler-Lagrange equations:

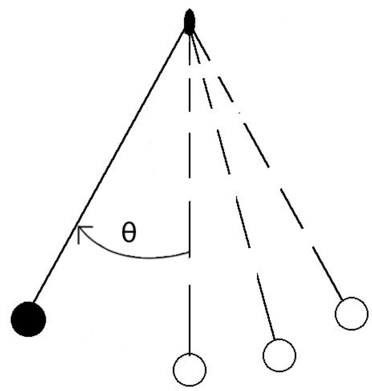
$$\boxed{\frac{\partial L}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)}$$

where i ranges from 1 to $3n-m$. Notice that these equations hold for <u>any</u> set of generalized coordinates! This allows us to exploit the symmetry of the problem in question by choosing the right set of generalized coordinates. An example will be considered after the next paragraph.

For a conservative system of one body, we got the Lagrangian $L = T - V = m\frac{\dot{x}^2}{2} - V(x)$. Thus we see that $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ is the momentum of the body, and similarly, $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F$ is the force exerted on the body. Consequently, we define $p^i = \frac{\partial L}{\partial \dot{q}^i}$ to be the *generalized momenta* and $Q^i = \frac{\partial L}{\partial q^i}$ to be the *generalized forces.* Notice that unlike the case of a conservative system, the generalized momenta and the generalized forces won't necessary coincide with the actual momentum of a body, or with the actual force exerted on it! The following table represents the analogue between the Lagrangian formulation as we did it earlier, to the Lagrangian formulation on a manifold.

Before	After: Lagrangian mecahnics on a manifold
No constraints	m holonomic constraints
Space is $\mathbb{R}$	Space is $\mathbb{R}^3$
Configuration space is $\mathbb{R}$	Configuration space is a $(3n-m)$-dim. manifold
Position x	Generalized coordinates $q = \left(q^1, \ldots, q^{3n-m}\right)$
Velocity $v = \dot{x}$	Generalized velocities $\dot{q} = \left(\dot{q}^1, \ldots, \dot{q}^{3n-m}\right)$
Momentum $p = m\dot{x} = \frac{\partial L}{\partial \dot{x}}$	Generalized momenta $p^i = \frac{\partial L}{\partial \dot{q}^i}$
Force $F = \frac{\partial L}{\partial x}$	Generalized forces $Q^i = \frac{\partial L}{\partial q^i}$
EL equation $\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$	EL equations $\frac{\partial L}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)$

Example. <u>A planar pendulum</u>: Due to the <u>holonomic</u> constraint imposed by the length of the wire l, the pendulum is constrainted to move on a circle. Thus the configuration space is the circle $\mathbb{S}^1$, which is a 1-dimensional manifold. As a 1-dimensional manifold, this system has one degree of freedom. We can take the generalized coordinate to be the angle $\theta(t)$ created by the pendulum. That is, if the length of the wire is l, we may write the position of the ball as $\vec{x} = (l\cos\theta, l\sin\theta)$.



The velocity is the derivative of the position, namely,

$$\dot{\vec{x}} = (-l\sin\theta, l\cos\theta)\,\dot{\theta}$$

We have, $\dot{\vec{x}}\cdot\dot{\vec{x}} = l^2\dot{\theta}^2$, so the kinetic energy is

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

The potential energy is $V = mgh$ where h denotes the height of the ball above its equilibruim position, and is $h = l\,(1-\cos\theta)$. Thus the potential energy is:

$$V = mgl\,(1-\cos\theta)$$

And the Lagrangian is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1-\cos\theta)$$

We only need to solve one Euler-Lagrange equation for the generalized coordinate θ (as it is a 1-dimensional configuration space). $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$ and $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$. Therefore Euler-Lagrange equation yields

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

Canceling ml^2 from both sides of the equation, we have,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

If we assume only a small change in the angle, namely $\theta \ll 1$, $\sin\theta \simeq \theta$ in first order and we get:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Setting $\omega = \sqrt{\frac{g}{l}}$ we get the same solution we get earlier for the harmonic oscillator, namely:

$$\theta\,(t) = A\sin\,(\omega t + \phi)$$

2.4. **Nonconservative forces.** One might notice that throughout our analysis of dynamical systems, we kept assuming that all forces are conservative. Well, yeah, many forces are conservative, but what about forces that are not conservative? For example, the electromagnetic forces on a moving charge depend on the velocity of the charge, so they will not be conservative. A resolution to this problem is provided by the notion of a *generalized potential*. Euler-Lagrange equations still work even if there is no potential function, V, in the usual sense, providing that the generalized forces are obtained from a function $U : TM \to \mathbb{R}$, $U(q, \dot{q})$ by the prescription[10]:

$$Q^i = -\frac{\partial U}{\partial q^i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}^i}\right)$$

In such case, Euler-Lagrange equations still hold, if we take the Lagrangian to be

$$L = T - U$$

Consequently, U is called a *generalized potential*. Systems for which all forces (except the forces of constraints) are derivable from generalized potential are called *monogenic*. Notice that if U is indepenet of the generalized velocities $\dot{q}$, the generalized potential reduces back to our previous notion of a potential function. Thus all conservative systems are monogenic.

Example. The electromagnetic field

Consider an electric charge, q, of mass m moving at a velocity, $\vec{v}$, in a charge-fre region containing both an electric field, $\vec{E}$, and a magnetic field, $\vec{B}$, whcih may depend upon time and position. The force the charge experiences is the Lorentz force, given by:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

We know from Maxwell's equation that $\vec{E}$ and $\vec{B}$ are derivable from a scalar potential ϕ and a vector potential $\vec{A}$ by[11]

$$\begin{aligned} \vec{E} &= -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \nabla \times \vec{A} \end{aligned}$$

If we take our velocity-dependent potential energy to be:

$$U = q\phi - q\vec{A} \cdot \vec{v}$$

the Lagrangian, $L = T - U$, is

$$L = \frac{1}{2}mv^2 - q\phi + q\vec{A} \cdot \vec{v}$$

And the x-component, for example, of Euler-Lagrange equations gives:

$$-q\frac{\partial \phi}{\partial x} + q\frac{\partial \vec{A}}{\partial x} \cdot \vec{v} = \frac{d}{dt}\left(m\dot{x} + qA_x\right)$$

which is equivalent to:

$$m\ddot{x} = q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) - q\left(\frac{\partial \phi}{\partial x} + \frac{dA_x}{dt}\right)$$

[10] But what are the generalized forces in such a case? We defined the generalized forces in terms of the Lagrangian, and the Lagrangian in terms of the potential energy (which is used then again in the generalized coordinates). It seems like the definitions are somehow recursive. So we need to find a way to define the generalized forces independently of the Lagrangian. The idea is to define generalized forces using D'Alembert's notion of virtual work. Since it is out of the scope of this paper, we will summarize by saying that the generalized forces can be defined to be $Q^i = \sum_j F^j \frac{\partial x^i}{\partial q^j}$, where F^i are the (not generalized) forces, x^i are the coordinates of the position and q^i are the generalized coordinates. Notice that if we take the generalized coordinates q^i to be our usual coordinates x^i, the generalized forces are simply the forces of the system. That is the case in most applications.

[11] The units are taken so that the speed of light is $c = 1$

But the total time derivative of A_x is

$$\begin{aligned}\frac{dA_x}{dt} &= \frac{\partial A_x}{\partial t} + \vec{v}\cdot\nabla A_x\\ &= \frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\end{aligned}$$

Combining the last two results, we obtain:

$$m\ddot{x} = q\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) + q\left(v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right)$$

which is just a very noneconomical way of writing:

$$m\ddot{x} = qE_x + q\left(\vec{v}\times\vec{B}\right)_x$$

Comparing the other components in a similar way, we see that the Lorentz force equation is derivable by means of the Euler-Lagrangian equations as well. In a similar way frictional forces and many other nonconservative forces can be taken care of.

2.5. **What are the advantages of the Lagrangian formulation?** My electrodynamics teacher used to say that the main advantage of the Lagrangian formulation is that Lagrange got an equation and a whole theory named after him... But in a more serious tone, not only that the variational principle formulation is a very elegant way of stating the already known laws of mechanics, it has another serious advantage: Unification. The Lagrangian formulation can be easily extended to describe systems that are not considered in Newtonian dynamics, such as the electromagnetic field, the Schrödinger equation in quantum mechanics, the standard model for particle physics and Einstein's equation for curved spacetime. Thus, the Lagrangian formulation provides a framework for theoretical extensions of many areas of physics, and most theories of physics can be described by using one single principle - the variational principle of the Lagrangian formulation.

References

[Frankel(2003)] Theodore Frankel. *The Geometry of Physics*. Cambridge University Press, 2003.

[Herbert Goldstein(2001)] John L. Safko Herbert Goldstein, Charles P. Poole. *Classical Mechanics*. Addison Wesley, 2001.

[V. I. Arnold(1997)] K. Vogtmann V. I. Arnold, A. Weinstein. *Mathematical Methods of Classical Mechanics*. Springer, 1997.