MATH 254: ORDINARY DIFFERENTIAL EQUATIONS

1. Introduction

Until now, you usually studied algebraic equations, such as: $x^2 - 2x + 1 = 0$. In this case the unknown is x, and its solutions are numbers. Differential equations are different. The unknown quantity in a differential equation is a function (that can represent many different things). A differential equation is an equation that involves derivatives of the unknown function. Recall that the derivatives are the rate of change of a quantity, so a differential equation is an equation that relates the rate of change of a certain quantity to the quantity itself.

1.1. **Examples.**

- Hooke's law: $m\ddot{x} = -kx$. Here x is in fact a function of time $x(t)$. The function x is called the dependent variable (as it depends on time t) and t is called the independent variable.
- Newton's law of universal graviation: $m\ddot{x} = -\frac{GMm}{x^2}$.
- Newton's second law: $m\ddot{x} = F$. In most cases, the force F is a function of the position x and the velocity $v = \dot{x}$ of the particle. So $m\ddot{x} = F(x, \dot{x})$.

The order of a differential equation is the order of the highest derivative it contains. So the last three examples are all second order ordinary differential equations.

- The simplest differential equation is $\frac{dy}{dx}$=0. Its solution is $y = const$. Notice that here y is the unknown quantity and it is a function of x. This is a first order differential equations.
- Radioactive decay of a material (with amout A) is modeled by the equation $\frac{dA}{dt} = -kA$ where $k > 0$ is a constant. This is a first order differential equations.
- Vibrating string is modeled by the equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. In this case there are partial derivatives, and the independent quantity is a function of time t and space x, so $u = u(t, x)$. This is a second order partial differential equations.

We will only concentrate on ordinary differential equations (ODE) in this class, and not partial differential equations (PDE).

There are other ways of classifying differential equations. One type of equation is a linear equation, it is an equation that depends linearly on the dependent variable. For example:

- The equation $\frac{d^2y}{dx^2} + y = 0$ is a linear equation, but the equation $\frac{d^2y}{dx^2} + y^3 = 0$ is a nonlinear equation.
- Don't get confused, the equation $\frac{d^2y}{dx^2} + y = x^3$ is a linear equation! This is because it is linear in y (the term x^3 is nonlinear, but it is the independent variable so it doesn't affect whether the equation is linear or not)

More examples:

- $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x^3$: Second-order, x independent, y dependent.

- $\sqrt{1 - \frac{d^2y}{dt^2}} - y = 0$: Second-order, t independent, y dependent.
- $\frac{d^4x}{dt^4} = xt$: Fourth-order, t independent, x dependent.

1.2. **Solutions of ODEs.**

- The function $y(x) = x^2 - x^{-1}$ is a solution of the linear equation $\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0$.
- For any choice of constants c_1, c_2 the function $y(x) = c_1e^{-x} + c_2e^{2x}$ is a solution of $y'' - y' - 2y = 0$.
- There are also implicit solutions, for example the equation $8x - 2y\frac{dy}{dx} = 0$ is solved implicitly by the relation $4x^2 - y^2 = C$. It is still called a solution because we got rid of all derivatives in the equation.

Notice that the second equation had two constant (and it was a second order equation). The third equation had one constant, and it is a first order equation. The general solution of an n-th order equation has n constants (that come from n integrations we have to make to solve the equation).

In real life situations the constants are usually known. For example, let's say that we want to solve the equation $\dot{y} = \sin 2t$. Integrating we see that $y = -\frac{1}{2}\cos 2t + c$. If we know the value of the function at a certain time (usually called the initial time), we can find the value of the constant. For example, the equation can be supplemented with the initial condition $y(0) = 0$. Then $-\frac{1}{2} + c = 0$ and $c = 1/2$. The problem of solving the ODE together with the IC (initial condition) is called an initial value problem. An nth order ODE should have n initial conditions at a given point (more about this later on).

2. 1st Order Equations

In this chapter we study first order equations, that can generally be written in the form $\frac{dy}{dx} = f(x, y)$.

2.1. Direction Fields. Draw and explain the direction fields of:

- $y' = 0$
- $y' = 1$
- $y' = y$

Show Wolfram demonstration. Explain in terms of the initial value formulation for:

- $y' = 1$, $y(0) = 1$.
- $y' = y$, $y(0) = 0$.
- $y' = y$, $y(0) = 1$.
- $\frac{dA}{dt} = -kA$ for $k > 0$. Explain physics of this decay model a bit.

2.2. Separable Equations. A general first order equation can also be written in the form $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ (for example by setting $M = -f, N = 1$). A separable equation is an equation for which M only depends on x and N only depends on y, namely: $M(x) + N(y)\frac{dy}{dx} = 0$. Such equation is called <u>separable</u> because it can be solved easily by separating the two variables. We write the equation as $M(x)dx = -N(y)dy$ and integrate (the left-hand side w.r.t x and the right-hand side w.r.t y). By the way, it is not always legitimate to multiply by a differential such as dx, but in this case, it's possible to proof that it's fine so we just won't worry about it. Let's see some examples:

- Initial Value Problem: $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$
  We can rewrite it in a separable form as $2(y - 1)dy = (3x^2 + 4x + 2)dx$, integrating gives $y^2 - 2y = x^3 + 2x^2 + 2x + c$, and the initial condition gives $c = 3$. This is of course an implicit solution. It can be turned into an explicit solution by solving for y.
- Initial Value Problem: $\frac{dy}{dx} = \frac{y \cos x}{1+2y^2}$, $y(0) = 1$
  We write it as $\frac{1+2y^2}{y}dy = \cos x dx$, and integrate to get $\ln |y| + y^2 = \sin x + 1$ using the initial condition. This is again an implicit solution.
- $\frac{dy}{dx} = y$, we continue as before to write $\frac{dy}{y} = dx$. Here we cannot divide by y if $y = 0$, which is a solution that we need to remember. Integrating we see that $\ln |y| = x + C$, or $y = Ce^x$. Notice that the case $C = 0$ corresponds to the other solution we found before.
- $\frac{dy}{dx} = \frac{y-1}{x+3}$, $y(-1) = 0$ gives $y(x) = -\frac{1}{2}(x + 1)$ in the same way.

2.3. Linear Equations. A linear equation is such that $f(x, y)$ is linear in y, so we can write $\frac{dy}{dx} + p(x)y = g(x)$. How can we solve such an equation?

Ex: let's start with the simple case $y' + ay = 0$. In this case we can see by inspection that y is a function whose derivative is $-a$ times the original function, so we can guess $y(x) = e^{-ax}$. More generally, the solution is $y(x) = Ce^{-ax}$. What if now, we want to solve: $y' + ay = g(x)$? In this case, if we can identify the left-hand side as the derivative of something, then the equation gets the form $\frac{d}{dx}(***) = g(x)$, and can be integrated immediately. To understand how this can be done, consider the solution we found before $y = Ce^{-ax}$. If we write it in the form $ye^{ax} = C$, we see that $\frac{d}{dx}(ye^{ax}) = 0$, which is exactly $e^{ax}(\frac{dy}{dx} + ay) = 0$. This is nothing other than the original equation but multiplied by e^{ax}.

Therefore we may write $e^{ax}y' + ae^{ax}y = e^{ax}g(x)$, or $\frac{d}{dx}(e^{ax}y) = e^{ax}g(x)$, and if $g(x)$ is known this can be integrated to give $y(x) = e^{-ax}\int^x e^{at}g(t)dt + ce^{-ax}$.

Let's see an actual example: $y' + 2y = e^{-x}$, with $y(0) = 3/4$. We multiply by e^{2x}, and continue to obtain $y = e^{-x} - \frac{1}{4}e^{-2x}$.

The general scheme. Consider $y' + p(x)y = g(x)$. We want to find a function $\mu(x)$ (an integrating factor) such that $\mu(x)[y' + p(x)y] = [\mu(x)y]'$. Expanding the right-hand-side, we see this is only possible if $\mu'(x)/\mu(x) = p(x)$ (for $\mu > 0$), so this function is nothing other than $\mu(x) = \exp\int^x p(t)dt$ (and μ is indeed positive. Since the integral of p is determined up to an additive constant, μ is determined up to a multiplicative constant and the sign is insignifcant.) Therefore $\mu(x)y = \int^x \mu(t)g(t)dt + c$ and this gives us y as being $y(x) = \frac{1}{\mu(x)}[\int^x \mu(t)g(t)dt + c]$.

More example:

- $y' - 2xy = x$, $y(0) = 0$. We want an integrating factor $\mu(x) = \exp - \int^x 2tdt = e^{-x^2}$. So $(e^{-x^2}y)' = xe^{-x^2}$ and by integrating we see that $y = -1/2 + ce^{x^2}$. The initial condition gives $c = 1/2$.
- $y' + 3y = x + e^{-2x}$
- $y' + y = xe^{-x} + 1$
- $y' - y = 2e^x$
- $\frac{1}{x}\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x$, $x > 0$. Here the integrating factor is x^{-2}, which gives a solution $y = x^2\sin x + Cx^2$.

Existence and Uniqueness. The most important application of differential equations is to use them to predict to future of whatever quantity they represent. As we said, this usually means that we need to know the initial value of the quantity at the present. This is not always possible, namely, there are differential equations with which we can use to predict the future. In this case of linear equations, there are conditions under which a unique solution exists:

Theorem 1. *Uniqueness and Existence Theorem: if the functions g and p are continuous on an open interval containing the point $x = x_0$, then there exists a unique solution $y = y(x)$ for the initial value problem $y' + p(x)y = g(x)$, $y(x_0) = y_0$.*

Although this might seem trivial, it is not the case. The following are nonlinear pathological examples:

- $y' = y^{1/3}, y(0) = 0$.
  Using separation of variables, we see that $y = (\frac{2}{3}x)^{3/2}$ solves the problem for $x \geq 0$. On the other hand, also $y = -(\frac{2}{3}x)^{3/2}$ solves it for $x \geq 0$. In fact, also $y(x) = 0$ is a solution...
  It is possible to show that in fact this problem has infinitely many solutions, and is therefore useless as a physical model. If the initial value doesn't lie on the x axis, one can show that there is a unique solution.
- $y' = y^2, y(0) = 1$. A solution to this problem is $y = \frac{1}{1-x}$. Clearly, this solution is undefined as $x \to 1$ and is therefore only valid for $x < 1$ (there is no indication for this feature in the ODE itself). Normally, the singularities appear whenever there are factors containg x that vanish in the equation.

2.4. **Exact Equations.** Consider the equation $\psi(x, y) = c$. Differentiate with respect to x (assuming $y = y(x)$) gives the ODE $\psi_x(x, y) + \psi_y(x, y)y' = 0$. Conversely, gives the ODE $M(x, y) + N(x, y)y' = 0$, when can go back to the first equation we wrote? This will of course be an implicit solution. For that, we need $\psi_x = M$ and $\psi_y = N$ to be satisfied. An equation that satisfies that, can be written as $\frac{d}{dx}[\psi(x, y(x))] = 0$ and is therefore called an exact differential equation. In practice, people often write the equations as $M(x, y)dx + N(x, y)dy = 0$.

Examples:

- $2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$ is exact, since $\frac{d}{dx}(x^2y^3) = 0$. Therefore $y = kx^{-2/3}$ is a solution for $x \neq 0$. $k = 0$ corresponds to the solution $y = 0$.
- $\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$. This equation can be written as $(2xy^2+1)dx+2x^2ydy = 0$, $\frac{2xy^2+1}{2x^2y}dx + dy = 0$, $dx + \frac{2x^2y}{2xy^2+1}dy = 0$, etc... What makes it easier to solve it? How can we know how we should write it to have it as an exact equation?

Theorem 2. *(Test for exactness) If the partials of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle, then $M(x, y)dx + N(x, y)dy = 0$ is exact in the rectangle if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.*

Back to the examples:

- We see that in our case the first equation $(2xy^2+1)dx+2x^2ydy = 0$ satisfies the test, and then $x^2y^2 + x = const$ is the solution.
- $(y \cos x+2xe^y)+(\sin x+x^2e^y-1)y' = 0$ has a solution $y \sin x+x^2e^y-y = c$
- $(2xy-\sec^2 x)dx+(x^2+2y)dy = 0$ is exact with solution $x^2y-\tan x+y^2 = C$
- $(3x^2 + 2xy) + (x + y^2)y' = 0$ is not exact, try to find a solution in the method described above, and you'll see it's impossible.

Integrating Factors. Sometimes, even when the equation $M(x, y)dx+N(x, y)dy = 0$ is not exact, we can try to choose a function μ such that $\mu M dx+\mu N dy = 0$ is exact. The condition for this is of course that $(\mu M)_y = (\mu M)_x$. Unfortunately, there is no general way to determine an integration factor, this can sometimes be as difficult as solving the original ODE. Normally, we make an Ansatz for it and check if it works. Notice that the integrating factor satisfies $M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$, so if it is a function of x only, we see that $\frac{d\mu}{dx} = \frac{M_y-N_x}{N}\mu$. Similar argument can work for μ as a function of y only. Examples:

- $(y^2 + xy)dx - x^2dy = 0$ is not exact, but you can check that $\mu = (xy^2)^{-1}$ is an integrating factor that gives the solution $\ln|x| + x/y = c$ for $x, y \neq 0$. $y = 0$ is another separate solution.
- $(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0$ is not exact, but its integrating factor is x^{-1} gives an implicit solution $x + x^3 \sin y = C$.
- $(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$ is not exact, but notice that $\frac{M_y-N_x}{N} = \frac{1}{x}$, so that $\frac{d\mu}{dx} = \frac{\mu}{x}$ hence $\mu = x$. This gives the solution $x^3y + \frac{1}{2}x^2y^2 = const$. In this problem the function $\mu = \frac{1}{xy(2x+y)}$ is also an integrating factor.

2.5. **Homogeneous equation.** Consider the following equation $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$. It is not separable, linear or exact. There also isn't any obvious integrating factor. It leads to another method for solving ODEs, change of variables. Change of variables may simplify the equation to an equation that is easier to solve. In this chapter, we'll look at first order homogeneous equations: $\frac{dy}{dx} = F(\frac{y}{x})$. Such equations can be solved by the change of variables $v = y/x$, $(y = xv)$ and then $\frac{dy}{dx} = v + x\frac{dv}{dx}$, or $\frac{dx}{x} = \frac{dv}{F(v)-v}$. Namely, it's always possible to separate a homogeneous equation.

Note however that it's not always possible to evaluate the integral on the right-hand-side. Examples:

- $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$ This is a homogeneous equation since we can write $\frac{dy}{dx} = (\frac{y}{x})^2 + 2\frac{y}{x}$. After the change of variable we get $\frac{dx}{x} = \frac{dv}{v(v+1)}$. Using partial sums, $\frac{dx}{x} = (\frac{1}{v} - \frac{1}{v+1})dv$ so $cx = \frac{v}{v+1}$. Substituting $y = vx$ back, we get $cx = \frac{y}{y+x}$, or $y = \frac{cx^2}{1-cx}$.

More Problems.

- $y' + \frac{1}{x}y = 3\cos 2x$, $x > 0$ (linear).
- $xy' + 2y = \sin x$, $x > 0$ (linear).
- $(1+x^2)y' + 4xy = (1+x^2)^{-2}$. (linear)
- $xy' + 2y = x^2 - x + 1, y(1) = 1/2$. (linear)
- $y' = -x/y \Rightarrow x^2 + y^2 = c^2$ (separation of variables). Solutions are circles.
- $y' = \frac{x-e^{-x}}{y+e^y}$ (separable)
- $y' = 1 + x + y^2 + xy^2$ (separable)
- $\frac{dr}{d\theta} = r^2/\theta$, $r(1) = 2$ (separable)
- $(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$ (exact)
- $\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}} = 0$ (exact)
- $(y/x + 6x)dx + (\ln x - 2)dy = 0$, $x > 0$ (exact)
- $x^2y^3 + x(1+y^2)y' = 0$, $\mu(x,y) = 1/xy^3$ is an integrating factor
- $ydx + (2x - ye^y)dy = 0$, $\mu(x,y) = y$ is an integrating factor
- $(x+2)\sin ydx + x\cos ydy = 0$, $\mu(x,y) = xe^x$ is an integrating factor.
- $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$ (homogeneous)
- $2ydx - xdy = 0$ (homogeneous)
- $(x^2 + 3xy + y^2)dx - x^2dy = 0$ (homogeneous)

2.6. **Applications and Modeling Problems.**

2.6.1. *Physics.* Newtonian Mechanics:
Newton's 2nd law $F = ma$, relates the acceleration of the body to the forces acting on it. Here the force is in Newton, the mass in kilograms and the acceleration in meters per second squared.
In the case of the force of gravity, the force is given by Newton's inverse-square law of gravitational attraction. If R is the radius of the earth, and x is the altitude above sea level, then $F = -\frac{GMm}{(R+x)^2} = -\frac{GMm}{R^2}\frac{1}{(1+\frac{x}{R})^2}$. We denote $g = GM/R^2 = 9.81\frac{m}{s^2}$, and notice that for $x/r \ll 1$, we can write $F = -mg(1 - 2\frac{x}{R} + \dots)$. So in the vicinity of the earth's surface, bodies obay the following equation $ma = mg$, or $\ddot{x} = g$. Notice that so far we neglet frictional forces (like the resistance due to air).

- Throwing a ball in the air (neglecting air resistance), from the earth surface at a velocity of v_0 m/s: We know $\ddot{x} = -g$, so $x(t) = -\frac{1}{2}gt^2 + v_0t$. Show Galileo's experiment.
- Motion of a parachuter: $m\frac{dv}{dt} = mg - kv$, or $\frac{dv}{dt} + \frac{k}{m}v = g$. This is a linear equation with an integrating factor $exp(kt/m)$. The solution is $v = \frac{mg}{k} + c_1e^{-kt/m}$. If the parachuter starts at rest, $v(0) = 0$ and $v = \frac{mg}{k}(1-e^{-kt/m})$. To obtain the position of the body, we replace v by dx/dt and integrate (assuming $x(0) = 0$) to obtain $x(t) = \frac{mg}{k}t - \frac{m^2g}{k^2}(1 - e^{-kt/m})$. Notice that as $t \to \infty$ we have $v_l = mg/k$. Namely a $75kg$ parachutist with air

resistance $15Nsec/m$ (when the chute is closed) can get to a maximum speed of $v_l = 49.05m/s$. This is about $180km/h$. Close chute can be modeled by a different friction coeffiction $k = 105Nsec/m$. Show Wolfram demonstration.

- Radiocarbon dating is a process used by anthropologists and archaeologists to estimate the age of organic matter (such as wood or bone). The vast majority of carbon on earth is nonradioactive carbon-12 (12C). However, cosmic rays cause the formation of carbon-14 (14C), a radioactive isotope of carbon which becomes incorporated into living plants (and therefore into animals) through the intake of radioactive carbon dioxide (14CO2). When the plant or animal dies, it ceases its intake of carbon-14, and the amount present at the time of death begins to decrease (since the14C decays and is not replenished). Since the half-life of 14C is known to be 5730 years, by measuring the concentration o f14C in a sample, its age can be determined. Radioactive matter disintegrates at a rate proportional to the amount present. If $Q(t)$ is the amount of it at time t, it therefore obays $\frac{dQ}{dt} = -rQ$ where $r > 0$ is the constant representing decay rate. Solving, we see that $Q(t) = Q_0 e^{-rt}$, and it's half time is therefore given by $\frac{1}{2} = e^{-r\tau}$ or $r = \frac{\ln 2}{\tau}$. In our case, $\tau = 5730$ so $r = \frac{\ln 2}{5730}$.
  If a fragment of bone is found with 20% of its original 14C, we have $0.2Q_0 = Q_0 e^{-rt}$, so $t = 13,300$ years is the age of the bone.
- Determining time of Death: Newton's law of cooling state that the rate at which a temperature of an object change is proportional to the difference between its temperature and the temperature of the surrounding. in other words, if $\theta(t)$ is the temperature of the object, then $\frac{d\theta}{dt} = -k(\theta - T)$ where T is the constant temperature of the surrounding and we assume $k > 0$ (with a minus sign in the equation) to represent the fact that the object is warmer than its surrounding.
  Assume we found a corpse at $t = 0$ with temerature 85F, and two hours later its temperature is 74F (with ambient temperature of 68F). The equation is separable so $\frac{d\theta}{\theta - T} = -kdt$ or $\ln|\theta - T| = -kt + C$ giving $\theta = T + (\theta_0 - T)e^{-kt}$ where we assume θ_0 to be the initial temperature.
  It is more convenient to write $\theta - T = (\theta_0 - T)e^{-kt}$ or $k = -\frac{1}{t}\ln\frac{\theta - T}{\theta_0 - T}$ ($t = -\frac{1}{k}\ln\frac{\theta - T}{\theta_0 - T}$) In our case, $k = -\frac{1}{2}\ln\frac{74-68}{85-68} = 0.52hr^{-1}$. So $t = -\frac{1}{0.52}\ln\frac{98.6-68}{85-68} = -1.129hours$ since $98.6F$ is the temperature of a living human. Therefore the body was discovered 1 hour and 8 minutes after death.

2.6.2. *Biology.*

- Exponential Growth: Consider $N(t)$ as the population of a given species at time t. The simplest model is that the rate of change of N is proportional to the current value of N.
  Let b be the number of births per unit time per elements of the specie, and d be the number of deaths per unit time per unit specie. Then $dN = (b-d)Ndt$, or if $r = b - d$ is a constant, $\frac{dN}{dt} = rN$. In this case, the constant r represents the rate of growth (if $r > 0$) or decay ($r < 0$). The solution is $N(t) = N_0 e^{rt}$. Show Wolfram.

- The previous model was observed to be reasonably accurate for many populations. However, eventually, limitations on space, food supply or other resources will reduce the growth rate and bring an end to the exponential growth. A more general model is of logistic growth, writing $dN/dt = f(N)N$ where $f(N)$ is a function such that $f(N) \sim r$ for small N, but gets smaller as N is sufficiently large. An example of such a model is $dN/dt = (r - aN)N$. It is one of the projects that can be studied.

2.6.3. *Economics.*